ERNEST M. HENLEY

ELEMENTARY QUANTUM FIELD THEORY

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WALTER THIRRING

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Preface

The aim of this book is to present that aspect of quantum field theory which is not obscured by mathematical difficulties and which does not require a deep understanding of special relativity. Within this scope the emphasis has been placed on particle physics rather than on other applications of quantum field theory.

To make the book comprehensible to a wide range of readers, we have presupposed only a knowledge of nonrelativistic quantum mechanics. All other tools that are needed are developed in the text. Thus, in the first part both the mathematical and physical descriptions of a quantum field are introduced. The conceptual aspects of the field are stressed. However, only fields that obey Bose-Einstein statistics are examined. Observables, invariants of the field, and internal symmetries are discussed.

In the second part of the book further techniques are developed by considering the interactions of a quantum field with various static sources. Those problems that are known to have exact solutions, namely, the neutral scalar theory, the pair theory, and the Lee model, are treated from both classical and quantum-mechanical points of view.

In the third part both the mathematical tools and the physical insight acquired in earlier chapters are applied to low-energy pion physics. After describing a classical approach and various other methods that have been used to analyze the problem in the past, we turn to the one model that is not based on uncontrolled mathematical approximations, namely, the static model developed by Chew and Low. In terms of this model we attempt to give the reader an understanding of pion-nucleon scattering, the static properties of nucleons, electromagnetic phenomena, and nuclear forces.

In the past few years a relativistic approach, based on analytic properties of the scattering matrix, has been evolving for the treatment of interacting fields. Although this approach reduces to that which we use in the nonrelativistic limit of the pion-nucleon problem, it is a wealthier one and contains much more of the physical situation than does the static model. It will thus ultimately allow a comparison with detailed experimental results. Unfortunately, these developments necessitate considerably more involved calculations than those presented here, and it is not yet clear whether a complete theory underlies them. Although relativistic treatments should ultimately remove all the shortcomings of the models discussed herein, the nonrelativistic approach will remain the basic first step to master.

For a unified treatment of all the problems covered, it seemed advantageous to work in a single representation. To emphasize the correspondence between classical and quantum-mechanical viewpoints, we chose the Heisenberg representation.

We have endeavored to cover the ground within our scope reasonably thoroughly, stressing the intuitive meaning of the results. We realize that rigor and simplicity are complementary aspects of a theory and have therefore tried to keep a reasonable balance between these features. We have not attempted to include a complete list of references, but we have tried to indicate where the reader can obtain further information whenever we felt that this was necessary. For additional study of the subject we refer the reader to N. N. Bogoliubov and D. V. Shirkov, "Introduction to the Theory of Quantized Fields" (Interscience Publishers, a division of John Wiley & Sons, Inc., New York, 1959), J. Hamilton, "The Theory of Elementary Particles" (Oxford University Press, New York, 1959), and S. S. Schweber, "An Introduction to Relativistic Quantum Field Theory" (Row, Peterson & Company, Evanston, Ill., 1961).

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Ernest M. Henley
Walter Thirring

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Part One

FREE FIELDS

CHAPTER 1

Introduction

1.1. Relation of Quantum and Classical Field Theory. Quantum theory provides us with a set of rules which are supposed to be of unlimited generality. They can be applied to any system and will tell us how our classical concepts have to be modified and how quantum features arise in the system under consideration. The application of these rules to fields creates quantum field theory. Elementary quantum mechanics is not a consistent theory when combined with classical field theory. It was pointed out by Bohr and Rosenfeld[1] that inconsistencies arise unless the classical electromagnetic field is quantized. If this is not done, then, in principle, the uncertainty relation between a position and a momentum component of a particle (e.g., an electron) can be violated. The normal Schrödinger or Klein-Gordon wave functions ψ can also be regarded as classical matter fields and should therefore be subject to quantization. It is the latter type of fields with which we shall be mainly concerned in this book. As in ordinary quantum mechanics, the quantization of fields is linked to the classical theory by the correspondence principle. It appears that the elementary quantum excitations of fields behave like particles; this is the only description we know at present to be applicable to elementary particles as we find them in nature. Correspondingly, quantum field theory dominates our thinking about the fundamental features of matter.

In the following we shall give a brief discussion of harmonic motions and fields in classical physics. The concept of a field is very wide, embracing all physical quantities which depend on space and time, like temperature, electric potential, and density. The common property of these phenomena is that there is an equilibrium state, and linear equations already reflect important behavioral features, if the departure from equilibrium can be considered small. Systems which are governed by similar types of equations (elliptic, hyperbolic, etc.[1]) show the same dynamical behavior, although they may represent completely different physical situations. Quantum field theory deals with hyperbolic equations. Accordingly, we may, for a first orientation, consider the simplest system of this type, namely, a vibrating line of atoms. Furthermore, we shall see that in many respects there is little difference between a continuous and a discrete line. Hence we shall start with the latter because it is closer to classical mechanics.[2] In the following we shall concentrate on the formal aspects of the problem, assuming that the physical situation is familiar to the reader.

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[1] N. Bohr and L. Rosenfeld, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.*, **12**(8) (1933); and L. Rosenfeld, in W. Pauli (ed.), "Niels Bohr and the Development of Physics," p. 70, McGraw-Hill Book Company, Inc., New York, 1955.

1.2. Vibrating Line of Atoms. If, as shown in Fig. 1.1, q_n denotes the displacement of the nth atom of a line from its equilibrium position,

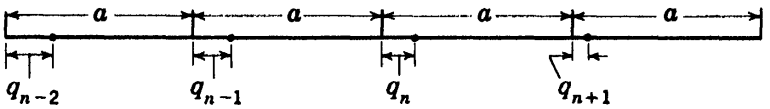

Fig. 1.1. Line of vibrating atoms. The equilibrium separation between atoms is a, and the instantaneous displacement of the nth atom is q_n.

$\dot{q}_n$ denotes the time derivative of this displacement, and we have harmonic forces between nearest neighbors, then the equation of motion is

$$\ddot{q}_n = \Omega^2(q_{n+1} + q_{n-1} - 2q_n) \tag{1.1}$$

Here we have given the atoms unit mass, and Ω^2 is the constant of the force between nearest neighbors. A macroscopic piece of the line is, of course, less rigid. If a line of N atoms is displaced by δx, then the restoring force is merely $\delta x\ \Omega^2/N$.

To make the system of coupled oscillators finite, we close the line after N atoms in such a manner that $q_{i+N} = q_i$. The problem posed by the N equations (1.1) can be solved by introducing normal coordinates. This is conveniently done with the aid of the Hamiltonian formalism, which will also be used later in the quantum theoretic development. The Hamiltonian (energy) of the line is readily seen to be ($p_n = \dot{q}_n$)

$$H = \sum_{n=1}^{N} \tfrac{1}{2}[p_n^2 + \Omega^2(q_n - q_{n+1})^2] \tag{1.2}$$

---

[1] See, e.g., H. Jeffreys and B. S. Jeffreys, "Methods of Mathematical Physics," p. 499, Cambridge University Press, New York, 1946.

[2] See, e.g., C. Kittel, "Introduction to Solid State Physics," 2d ed., p. 103, John Wiley & Sons, Inc., New York, 1953.

Hamilton's canonical equations

$$\dot{q}_n = \frac{\partial H}{\partial p_n} = p_n$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} = \Omega^2(q_{n+1} - q_n + q_{n-1} - q_n)$$

being equivalent to (1.1).

We define the normal coordinates Q_s and momenta P_s by

$$q_n = \sum_s e^{isn}\frac{Q_s}{N^{\frac{1}{2}}} \qquad p_n = \sum_s e^{-isn}\frac{P_s}{N^{\frac{1}{2}}} \tag{1.3}$$

where, in accordance with the periodic boundary condition introduced above, s takes on the values $s = 2\pi l/N$, l being an integer between $-N/2$ and $N/2$. Since the q_n are real, the Q_s and P_s themselves are complex and satisfy

$$Q_{-s} = Q_s^* \qquad P_{-s} = P_s^* \tag{1.4}$$

With the aid of the formula

$$\sum_{n=1}^{N} e^{in(s-s')} = N\delta_{s,s'} \tag{1.5}$$

we may also invert (1.3) to

$$Q_s = \sum_{n=1}^{N} e^{-isn}\frac{q_n}{N^{\frac{1}{2}}} \qquad P_s = \sum_{n=1}^{N} e^{isn}\frac{p_n}{N^{\frac{1}{2}}} \tag{1.6}$$

Inserting (1.3) into the Hamiltonian, we find, by means of (1.4) and (1.5),

$$H = \tfrac{1}{2}\sum_s\left[P_sP_s^* + Q_sQ_s^*\Omega^2\left(2\sin\frac{s}{2}\right)^2\right] \tag{1.7}$$

Thus the normal coordinates serve to uncouple the oscillators, and the equation of motion coming from the Hamiltonian (1.7) is simply[1]

$$\ddot{Q}_s = -\omega_s^2 Q_s \qquad \omega_s = 2\Omega\sin\frac{s}{2} \tag{1.8}$$

which also appears by inserting (1.3) directly into (1.1). The solution of (1.8) can be written as

$$Q_s(t) = Q_s(0)\cos\omega_s t + \frac{\dot{Q}_s(0)}{\omega_s}\sin\omega_s t$$

$$q_n(t) = N^{-1}\sum_{n',s}\left\{q_{n'}(0)\cos\left[s(n-n') - \omega_s t\right] + \frac{\dot{q}_{n'}(0)}{\omega_s}\sin\left[s(n-n') - \omega_s t\right]\right\}$$

---

[1] The reader may check for himself that Q_s and Q can$_s^*$ be treated as independent variables.

Equation (1.3) together with (1.8) tells us that the motion of our system is a superposition of vibrations with frequencies ω_s, many of which are much smaller than Ω ($\omega_s \sim \Omega 2\pi l/N$), corresponding to the smaller rigidity of the whole line, about which we remarked earlier. This well-known fact is, for instance, the point of Debye's theory of the specific heat, as opposed to Einstein's. In quantum mechanics we shall see that the excited states of the oscillators Q_s with energies ω_s behave in many respects like particles. Oscillations of the type considered here are very common phenomena, appearing as sound waves in solids and liquids, as spin waves in ferromagnetics, and as surface waves in nuclei. In all these cases we find the particlelike behavior of the elementary excitations of the oscillators Q_s with energies ω_s. In fact, the sound waves in liquid helium are the closest mechanical model of elementary particles that we have.

1.3. Continuous Vibrating Line. In many cases it is expedient to look at the situation from the macroscopic, rather than from the microscopic, point of view and not to resolve the line into individual atoms. This can be done by a limiting process in which $N \rightarrow \infty$ as the distance between the atoms $a \rightarrow 0$ but the length $L = aN$ of the line remains constant. This means, however, that the system now acquires infinitely many degrees of freedom and that we need this number of variables for its description.

Calling x the distance from the origin of the line[1] [$q_n \rightarrow q(x)$], we obtain for the equation of motion in this limit the familiar partial differential equation

$$\ddot{q}(x) = \Omega^2 a^2 \frac{\partial^2}{\partial x^2} q(x) \tag{1.9}$$

In (1.9) Ωa appears to be the wave velocity v, so that Ω must behave in this limit like $\Omega = v/a$. Thus, to obtain with infinitely many atoms a line of finite rigidity requires an infinite force between neighboring atoms. As a consequence, the line will be capable of vibrations with infinite frequencies.

The general solution of (1.9) subject to $q(x) = q(x + L)$ can again be obtained by carrying out the limiting process on the normal coordinates in (1.9). With

$$k = \frac{s}{a} = \frac{2\pi l}{L} \qquad -\infty < l < \infty$$

and

$$q(x) = \frac{1}{L^{1/2}} \sum_k e^{ikx} Q_k \tag{1.10}$$

---

[1] We actually put $q(x) = q_n/a^{\frac{1}{2}}$, so that, for a finite energy, $q(x)$ will remain finite.

and from the limit of (1.5),

$$\sum_{n=1}^{N} \to \frac{1}{a}\int_0^L dx \qquad \int_0^L dx\, e^{ix(k-k')} = L\delta_{k,k'}$$

we find from (1.9), for the equations of motion of Q_k,

$$\ddot{Q}_k = -k^2v^2Q_k \tag{1.11}$$

Thus the frequency ω_k is kv, which is the limit of our previous expression (1.8) for ω_s. Hence it is just for short wavelengths that the atomic structure of the chain transpires. Introduction of normal coordinates means solving a partial differential equation by a Fourier expansion. These results can also be deduced from the Hamiltonian (1.2), which becomes in the limit[1]

$$H = \tfrac{1}{2}\int_0^L dx\left[\dot{q}^2(x) + v^2\left(\frac{\partial q}{\partial x}\right)^2\right] = \tfrac{1}{2}\sum_k(|\dot{Q}_k|^2 + k^2v^2\,|Q_k|^2) \tag{1.12}$$

Whereas for the enumerable coordinates Q_k Eq. (1.12) leads directly to Eq. (1.11), we have to generalize the Hamiltonian formalism of ordinary mechanics to get the equations of motion for the nonenumerable coordinates $q(x)$. This can be done by introducing functional derivatives with the aid of Dirac's δ function:

$$\begin{aligned} \frac{\delta q(x)}{\delta q(x')} &= \delta(x - x' \\ \frac{\delta}{\delta q(x')}\frac{\partial q(x)}{\partial x} &= \frac{\partial}{\partial x}\delta(x - x') \end{aligned} \tag{1.13}$$

The former is the continuum form of $\partial q_n/\partial q_{n'} = \delta_{n,n'}$, and the latter is obtained by differentiating with respect to x.

Writing the Hamiltonian in terms of canonical variables p and q,

$$H = \tfrac{1}{2}\int_0^L dx\left\{p^2(x) + v^2\left[\frac{\partial q(x)}{\partial x}\right]^2\right\}$$

we find that Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

are generalized to

$$\begin{aligned} \dot{p}(x) &= -\frac{\delta H}{\delta q(x)} = -v^2\int_0^L dx'\,\frac{\partial q(x')}{\partial x'}\frac{\partial}{\partial x}\delta(x - x') = v^2\frac{\partial^2}{\partial x^2}q(x) \\ \dot{q}(x) &= \frac{\delta H}{\delta p(x)} = p(x) \end{aligned} \tag{1.14}$$

---

[1] We give the whole line unit mass.

and agree with (1.9). These formal tools will find frequent applications later, since we shall deal mainly with the continuous case, which is almost simpler than the atomistic point of view. It allows us to eliminate the microscopic constants Ω, a, N and to replace them by the macroscopic constants v, L.

Some fields, however, such as the electromagnetic one and those of elementary particles, do not possess mechanical backgrounds to serve as guides in writing equations of motion. We have to appeal to the special theory of relativity to obtain the invariance properties of these fields. The four-dimensional homogeneity and isotropy of our space-time continuum are supposed to emerge from the same property of the fields of all the elementary particles. In technical language, the invariance under the inhomogeneous Lorentz group is the only guiding principle which allows us to select the possible field equations for elementary particles. This daringly speculative procedure is, in fact, very successful and reveals many startling properties of elementary particles. Unfortunately, the theory of the representations of the Lorentz group is far from being elementary, so that we shall not be able to give a systematic discussion of relativistic field theory. However, we shall encounter the influence of relativity theory on our notions about particles.

The requirements of Lorentz invariance gives fields remarkable properties which are not possessed by any mechanical system. Since the theory has to be invariant under arbitrary space-time displacements, the field cannot have any atomic structure but must be continuous. Furthermore, the field must fill all space and time; it has to last forever everywhere and can never be removed. Thus we arrive at a new outlook on space and matter. Space is spanned by the continuous background of the fields of elementary particles; in some respects this is the sequel of the ether concept of the last century. Matter is just a local excitation of this background, something accidental. There is no conservation of matter, and the laws governing the interactions of matter are secondary and complex. The simplicity of nature is revealed by the equations of the elementary fields, which reflect symmetry and regularity. This is quite a different picture from the mechanical one, in which matter is supposed to be fundamental and the law of force between its constituents is the primary law of nature. This explains why the present fundamental research in physics makes so much use of quantum field theory, which concentrates on exploring the properties of this background for all physical phenomena.

CHAPTER 2

The Harmonic Oscillator

2.1. Eigenvalues of H. For the fields considered in Chap. 1 it was shown that the basic equations of motion are like those of a simple harmonic oscillator or of a set of coupled harmonic oscillators. In this chapter we shall therefore give the quantum theory of the harmonic oscillator in a form appropriate for later developments. It will appear in the following chapters that the quantum development for coupled oscillators and for fields is a straightforward generalization of the theory for this system with one degree of freedom. Moreover, the typical quantum features of fields are already encountered in rudimentary form in the harmonic oscillator, where they are familiar from elementary discussions.

Our problem is characterized by the Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) \tag{2.1}$$

The coordinate q and the momentum p are now operators which obey the commutation relation[1]

$$[q,p] = i \tag{2.2}$$

It is a typical prediction of quantum theory that measurement of an observable cannot yield an arbitrary result, but only an eigenvalue of the operator associated with this observable. We must therefore seek the eigenvalues of observables such as the energy (2.1). This problem can be attacked in several ways. For instance, we can satisfy (2.2) by representing p by the differential operator $-i\partial/\partial q$ and solve the differential equation arising from the eigenvalue problem $(H - E)\psi = 0$. Such an approach is not the shortest one, and for our purpose a purely algebraic method is more convenient. We introduce the operators which correspond to the amplitude of the classical motion:

---

[1] We shall always use appropriate units with $\hbar = 1$.

$$\begin{aligned} a &= \frac{\omega q + ip}{(2\omega)^{\frac{1}{2}}} \\ a^{\dagger} &= \frac{\omega q - ip}{(2\omega)^{\frac{1}{2}}} \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} q &= \frac{a + a^{\dagger}}{(2\omega)^{\frac{1}{2}}} \\ p &= -i\omega \frac{a - a^{\dagger}}{(2\omega)^{\frac{1}{2}}} \end{aligned} \tag{2.4}$$

It follows from (2.2) that the commutation relations for a and $a^{\dagger}$ are

$$\begin{aligned} [a,a] &= [a^{\dagger},a^{\dagger}] = 0 \\ [a,a^{\dagger}] &= 1 \end{aligned} \tag{2.5}$$

In terms of a and $a^{\dagger}$ the Hamiltonian becomes, by means of (2.4) and (2.5),

$$H = \omega(a^{\dagger}a + \tfrac{1}{2}) \tag{2.6}$$

Since the position and momentum operators q and p do not commute with the Hamiltonian, the operators a and $a^{\dagger}$ will not commute with it either. In fact, from (2.5) and (2.6) it follows that

$$\begin{aligned} [H,a^{\dagger}] &= \omega a^{\dagger} \\ [a,H] &= \omega a \end{aligned} \tag{2.7}$$

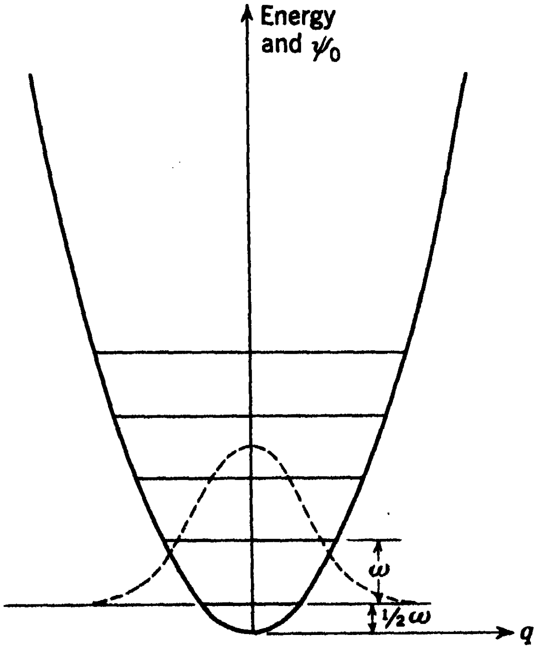

Fig. 2.1. Harmonic-oscillator potential with energy eigenvalues and ground-state wave function $\psi_0(q)$.

The form of the commutation relations (2.7) allows us to draw conclusions about the eigenvalues of H. Applying them to an eigenfunction ψ_E of H with eigenvalue E, $(H - E)\psi_E = 0$, we find

$$Ha\psi_E = (E - \omega)a\psi_E \tag{2.8}$$

This tells us that $a\psi_E$ is another eigenfunction of H belonging to the eigenvalue $E - \omega$. Similarly, it follows from the other relation (2.7) that $a^{\dagger}$ increases an eigenvalue by ω. This shows that there are equally spaced sequences of eigenvalues of H with the spacing ω. However, these sequences have to terminate somewhere, since $a^\dagger a$ is positive definite and H can, therefore, possess no eigenvalue <0. This condition requires that for a certain state ψ_0 the relation $a\psi_0 = 0$ hold, in which case we cannot get a lower eigenvalue by applying a. From (2.6) we see that ψ_0 is an eigenfunction of H and belongs to the eigenvalue $\omega/2$. Furthermore, our condition determines ψ_0 uniquely, so that there is only one sequence of eigenvalues. We can summarize our findings as follows. The eigenvalue spectrum of H is shown in Fig. 2.1 and is

$$E = n\omega + \tfrac{1}{2}\omega \tag{2.9}$$

n being a nonnegative integer.

2.2. Properties of the Eigenstates of H. The state ψ_0 corresponding to E_0 obeys

$$a\psi_0 = 0 \tag{2.10}$$

and the state ψ_n belonging to E_n is obtained by applying $a^\dagger$ n times to ψ_0,

$$\psi_n = \frac{a^{\dagger n}}{(n!)^{\frac{1}{2}}}\psi_0 \tag{2.11}$$

That $(n!)^{-\frac{1}{2}}$ is the correct normalization factor, provided ψ_0 is normalized, can be seen most easily by induction:

$$\int \psi_n^* \psi_n = \int \psi_{n-1}^* \frac{aa^\dagger}{n} \psi_{n-1} = \frac{1}{n}\int \psi_{n-1}^* \left(\frac{H}{\omega} + \frac{1}{2}\right)\psi_{n-1} = \int \psi_{n-1}^*\psi_{n-1} \tag{2.12}$$

Repeated use of this equation finally leads to $\int\psi_n^*\psi_n = \int\psi_0^*\psi_0$.

The present method can also be used to obtain an explicit representation of the eigenfunctions ψ_n in terms of the space variable q. Since in this representation $p = -i\partial/\partial q$, we have

$$a\psi_0(q) = \left(\frac{1}{2\omega}\right)^{\frac{1}{2}}\left(\omega q + \frac{\partial}{\partial q}\right)\psi_0(q) = 0$$

and it follows that the normalized ground-state eigenfunction is

$$\psi_0(q) = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\omega q^2/2} \tag{2.13}$$

The higher excited states ψ_n are then obtained by the application of the differential operator $a^\dagger = (\omega q - \partial/\partial q)(2\omega)^{-\frac{1}{2}}$.

The width of the ground-state wave function is of the order $\omega^{-\frac{1}{2}}$; this expresses the quantum-mechanical zero-point fluctuation Δq of the position q and is shown in Fig. 2.1. Formally, we can define this uncertainty by

$$(\Delta q)^2 = \int \psi_0^*(\bar{q} - q)^2 \psi_0 \, dq \tag{2.14}$$

where $\bar{q}$ is the mean value of q and is easily seen to vanish by means of (2.10) and its hermitian conjugate

$$\bar{q} = \int \psi_0^* q \psi_0 \, dq = \int \psi_0^*(a + a^\dagger)\psi_0 \frac{1}{(2\omega)^{\frac{1}{2}}} \, dq = 0 \tag{2.15}$$

This result is also apparent from $\psi_0(q) = \psi_0(-q)$. With our methods we find that

$$\begin{aligned}(\Delta q)^2 &= \int \psi_0^* q^2 \psi_0 \, dq = \int \psi_0^* \frac{aa^\dagger}{2\omega} \psi_0 \, dq \\ &= \frac{1}{2\omega} \int \psi_0^* [a,a^\dagger] \psi_0 \, dq = \frac{1}{2\omega}\end{aligned} \tag{2.16}$$

Physically, this is a direct consequence of the uncertainty principle $\Delta q \, \Delta p \geq \frac{1}{2}$. The lowest energy is not obtained by localizing the particle sharply at the origin, since this would entail a large Δp and hence kinetic energy. Since the mean values of the position and momentum are zero, the average value of the energy is $\frac{1}{2}[(\Delta p)^2 + \omega^2(\Delta q)^2]$. If this is minimized with respect to Δq, with the constraint that $\Delta p \, \Delta q = \frac{1}{2}$ we find that the minimum is $\omega/2$ for $\Delta q = (2\omega)^{-\frac{1}{2}}$. That is to say, the most energetically favorable compromise is close to the value for Δq given by (2.16), and the lowest energy is not zero, as in the classical case, but $\omega/2$.[†] These quantum-mechanical fluctuations are usually small but sometimes lead to macroscopic effects. For instance, they prevent liquid helium from solidifying under normal pressure.

2.3. Time Dependence of Motion. From these quantum features we now turn to dynamical aspects which reflect the classical harmonic motion of the system. The time development can be described in quantum mechanics in many ways.[1] We can, for instance, consider the operators constant and the state vector time-dependent according to

$$\psi(t) = e^{-iHt}\psi(0) \tag{2.17}$$

(Schrödinger representation). Another possibility is to consider the state vector constant and to apply the unitary operator e^{iHt} to the operators $\mathcal{O}$, so that they vary with time according to

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[†] It is only an accident that this rough argument with the uncertainty relation gives the exact numerical result. However, one usually gets the correct order of magnitude in this manner.

[1] See P. A. M. Dirac, "The Principles of Quantum Mechanics," 3d ed., chap. V, Oxford University Press, New York, 1947.

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \tag{2.18}$$

(Heisenberg representation). This obviously leads to the same expectation values,

$$\psi^*(t)\mathcal{O}(0)\psi(t) = \psi^*(0)\mathcal{O}(t)\psi(0)$$

and to the same physical consequences.

In the latter case the time dependence of the operators is such that they obey the classical equations of motion, since (2.18) yields

$$\dot{\mathcal{O}}(t) = i[H,\mathcal{O}(t)] \tag{2.19}$$

and the commutator gives the same expression as the Poisson bracket in classical mechanics. Because in our problems the classical equations will be of a well-known structure and tools for their solution are readily available, we shall always stay in the Heisenberg representation.[1] Furthermore, to reserve the letter ψ for the field operators, we shall use Dirac's[2] bra and ket notation from now on. In it the state vector and its complex conjugate are denoted by the brackets $|\rangle$ and $\langle|$. To specify the state vector, we may write some labels into the bracket. For instance, the energy eigenstates of the harmonic oscillator can simply be characterized by the associated quantum number of the state, e.g., $|n\rangle$, so that the Schrödinger equation reads

$$(H - E_n)\,|n\rangle = 0$$

In elementary wave mechanics this notation corresponds to denoting a vector by a single symbol $\mathbf{r}$ rather than specifying its components in a particular frame, e.g., x, y, z. The latter appear as the scalar product of the vector $\mathbf{r}$ with the unit vector $\mathbf{n}$ in the direction of the axes (in a particular frame) under consideration. Thus, $x = \mathbf{r}\cdot\mathbf{n}_x$. Correspondingly, the Schrödinger function $\psi_n(q)$ at a point in coordinate space, q', is the component of the state $|n\rangle$ in the direction of an eigenvector $|q'\rangle$ of the operator q and is given by the scalar product

$$\psi_n(q') = \langle q' \,|\, n\rangle$$

The components of the state $|n\rangle$ in another frame are linear combinations of $\psi_n(q')$ in the same way that the components of $\mathbf{r}$ are in a frame given by the unit vectors $\mathbf{n}_i$:

$$x' = \mathbf{r}\cdot\mathbf{n}_{x'} = \sum_{i=x,y,z} (\mathbf{r}\cdot\mathbf{n}_i)(\mathbf{n}_i\cdot\mathbf{n}_{x'}) = \sum_{i=x,y,z} r_i(\mathbf{n}_i\cdot\mathbf{n}_{x'})$$

---

[1] There are other useful representations, in which part of the time dependence is retained by the state functions. We shall not be concerned with these here.

[2] Dirac, *op. cit.*, chaps. I–III.

For instance, in momentum space, where we use the eigenstates $|\,p'\rangle$ of the operator p, we obtain, in analogy with the above,

$$\phi_n(p') = \langle p'\,|\,n\rangle = \int dq'\,\langle p'\,|\,q'\rangle\langle q'\,|\,n\rangle = \int dq'\,\langle p'\,|\,q'\rangle\psi_n(q')$$

which is the usual expression for the wave function in momentum space if we insert

$$\langle p'\,|\,q'\rangle = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-ip'q'}$$

Our new notation can be illustrated for a general operator $\mathcal{O}$ by analogy with the above development. Thus, an eigenvalue equation

$$\mathcal{O}\,|\,n\rangle = \alpha\,|\,n\rangle$$

can be rewritten, by multiplying by $\langle q'|$, as

$$\langle q'\,|\,\mathcal{O}\,|\,n\rangle = \int dq''\,\langle q'\,|\,\mathcal{O}\,|\,q''\rangle\langle q''\,|\,n\rangle = \alpha\langle q'\,|\,n\rangle$$

or, equivalently,

$$\int dq''\,\langle q'\,|\,\mathcal{O}\,|\,q''\rangle\psi_n(q'') = \alpha\psi_n(q')$$

The operator is therefore a matrix in a particular representation. Similarly, a general matrix element $\langle m\,|\,\mathcal{O}\,|\,n\rangle$ can be rewritten as

$$\langle m\,|\,\mathcal{O}\,|\,n\rangle = \int dq'\,dq''\,\psi_m^*(q')\langle q'\,|\,\mathcal{O}\,|\,q''\rangle\psi_n(q'')$$

These general equalities are considerably simplified if a representation is chosen in which $\mathcal{O}$ is a diagonal matrix,

$$\langle q'\,|\,\mathcal{O}\,|\,q''\rangle = \delta(q' - q'')\mathcal{O}(q'')$$

For example, we can find $\langle p'\,|\,q'\rangle$ as follows:

$$q\,|\,q'\rangle = q'\,|\,q'\rangle$$

$$\langle p'\,|\,q\,|\,q'\rangle = \int \langle p'\,|\,q\,|\,p''\rangle\langle p''\,|\,q'\rangle\,dp'' = q'\langle p'\,|\,q'\rangle$$

In momentum space, the diagonal representation of q is

$$\langle p'\,|\,q\,|\,p''\rangle = -i\delta'(p' - p'')$$

where δ' is the first derivative of the δ function. We therefore obtain[1]

$$i\frac{\partial}{\partial p'}\langle p'\,|\,q'\rangle = q'\langle p'\,|\,q'\rangle$$

$$\langle p'\,|\,q'\rangle \propto e^{-ip'q'}$$

---

[1] $\propto$ stands for "proportional to."

Having dealt with these preliminaries, we can study the motion of the system. According to (2.7) the equation of motion (2.19) for the operator a is simply

$$\dot{a}(t) = -i\omega a(t) \tag{2.20}$$

which can immediately be integrated to

$$a(t) = e^{-i\omega t}a(0) \tag{2.21}$$

From this result and its hermitian conjugate we find that the position and momentum are the same functions of time as in classical mechanics:

$$q(t) = \frac{a(0)e^{-i\omega t} + a^\dagger(0)e^{i\omega t}}{(2\omega)^{1/2}} \tag{2.22a}$$

$$p(t) = -\frac{i\omega[a(0)e^{-i\omega t} - a^\dagger(0)e^{i\omega t}]}{(2\omega)^{1/2}} \tag{2.22b}$$

To learn something about the time dependence of our system in a certain state $|\,\rangle$, we shall calculate

$$|\langle q'(t) \mid \rangle|^2 = |\psi(q',t)|^2$$

which represents the probability of finding the oscillator at q' at the time t. For the states $|\,n\rangle$ this will, of course, be time-independent; in our representation this follows from (2.21). To obtain something more interesting, we have to consider a superposition of the states $|\,n\rangle$. In particular, it is useful to study the state (wave packet) $|\,d\rangle$, which is an eigenstate of the nonhermitian operator a:

$$a(0) \mid d\rangle = \left(\frac{\omega}{2}\right)^{1/2} d \mid d\rangle \tag{2.23}$$

As we shall show, it undergoes harmonic motion of period equal to the classical frequency ω of the oscillator. In analogy with (2.13), we have

$$\psi_d(q') = \langle q' \mid d\rangle = \left(\frac{\omega}{\pi}\right)^{1/4} e^{-(q'-d)^2\omega/2} \tag{2.24}$$

e.g., a gaussian distribution of the same width as the ground state but displaced from the origin by the distance d. To be sure, such a state is not an eigenstate of H, and our formalism tells us immediately that the proportion of the state $|\,n\rangle$ present in the packet is

$$|\langle n \mid d\rangle|^2 = \frac{1}{n!}|\langle 0 \mid a^n \mid d\rangle|^2 = \left(\frac{\omega d^2}{2}\right)^n \frac{1}{n!}|\langle 0 \mid d\rangle|^2$$

Making use of the completeness of the set of states $|n\rangle$, we find

$$\sum_{n=0}^{\infty} |\langle n \mid d\rangle|^2 = e^{d^2\omega/2} |\langle 0 \mid d\rangle|^2 = 1$$

and

$$|\langle n \mid d\rangle|^2 = e^{-d^2\omega/2}\left(\frac{d^2\omega}{2}\right)^n \frac{1}{n!} \tag{2.25}$$

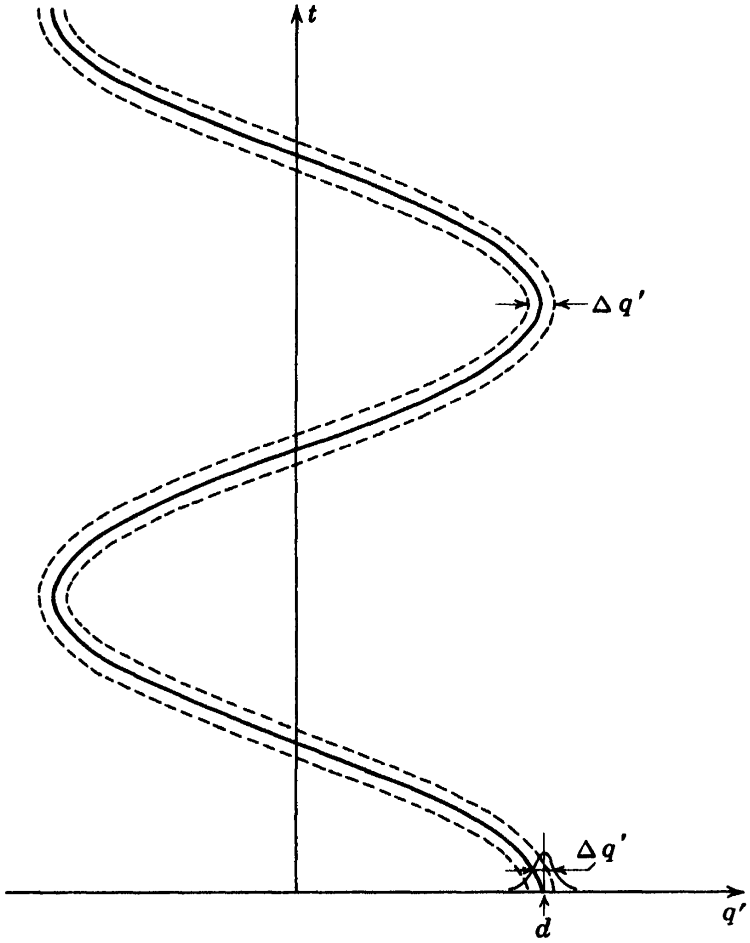

Fig. 2.2. Representation of motion of packet $|d\rangle$. The motion of the center of the packet and its width $\Delta q'$ are represented. The distribution of the packet is also shown at $t = 0$.

Thus the probability of finding the nth excited state in $|d\rangle$ follows a Poisson law[1] with a mean value of

$$n = \frac{\omega d^2}{2} \sim \left(\frac{\text{displacement}}{\text{zero-point fluctuation}}\right)^2$$

We shall see shortly that the wave packet performs harmonic motion with an amplitude d and frequency ω. Hence the dominant state in it is the excited level for which the energy equals $\frac{1}{2}\omega^2 d^2$, e.g., the classical energy of such a motion.

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[1] H. Margenau and G. M. Murphy, "The Mathematics of Physics and Chemistry," p. 425, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

In order to express $| d\rangle$ in terms of the eigenstates of $q(t)$, we remember from (2.22) that[1]

$$a = e^{i\omega t}\left(\frac{\omega}{2}\right)^{\frac{1}{2}}\left[q(t) + \frac{ip(t)}{\omega}\right] \tag{2.26}$$

Thus we find, for $\psi_d[q'(t)] = \langle q'(t) \mid d\rangle$,

$$\left[\omega q'(t) + \frac{\partial}{\partial q'(t)}\right]\psi_d = \omega\, d e^{-i\omega t}\psi_d \tag{2.27}$$

which integrates to

$$\psi_d[q'(t)] \propto \exp\left[-\frac{\omega}{2}(q' - de^{-i\omega t})^2\right]$$

or, on normalizing,

$$\psi_d[q'(t)] = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{\omega}{2}(q' - d\cos\omega t + 2id\sin\omega t)(q' - d\cos\omega t)\right]$$

$$|\psi_d[q'(t)]|^2 = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\omega[q'(t) - d\cos\omega t]^2} \tag{2.28}$$

This demonstrates that our wave packet performs rigid oscillations with frequency ω, as shown in Fig. 2.2. For the mean values of position and momentum, we get, from (2.28),

$$\langle d \mid q(t) \mid d\rangle = d\cos\omega t$$

$$\langle d \mid p(t) \mid d\rangle = -\omega d\sin\omega t$$

Hence the state $| d\rangle$ is the appropriate quantum-mechanical generalization for classical harmonic motion and will find frequent applications later.

In our subsequent studies of more complicated systems we shall follow the pattern of the above treatment for the harmonic oscillator, since we shall always encounter similar situations.

[1] Here and subsequently, when no time is shown after an operator, we shall imply that $t = 0$, e.g., $a = a(t = 0) = a(0)$.

CHAPTER 3

Coupled Oscillators

3.1. Eigenvalues of the Hamiltonian. To approach quantum field theory, we now treat a system of coupled oscillators quantum-mechanically. The problem we shall consider is a generalization of that dealt with in Chap. 1 (Fig. 1.1), namely, oscillators on a closed line coupled not only to their neighbors but also to their equilibrium position. The Hamiltonian and the equations of motion for such a system are expressed in terms of the generalized coordinates q_n and momenta p_n:

$$\begin{aligned} H &= \sum_{n=1}^{N} \tfrac{1}{2}[p_n^2 + \Omega^2(q_n - q_{n+1})^2 + \Omega_0^2 q_n^2] \\ \ddot{q}_n &= \Omega^2(q_{n+1} + q_{n-1} - 2q_n) - \Omega_0^2 q_n \end{aligned} \tag{3.1}$$

Putting the second independent frequency Ω_0 equal to zero would bring us back to (1.2). It will turn out later that, in the continuum limit, (1.2) corresponds to the case of a massless field, whereas (3.1) corresponds to a field with "mass" Ω_0. As in Chap. 2, we now have to state the commutation relations, which are, according to the general rules of quantum mechanics,[1]

$$\begin{aligned} [q_l, p_m] &= i\delta_{l,m} \\ [q_l, q_m] &= [p_l, p_m] = 0 \end{aligned} \tag{3.2}$$

To work out the eigenvalues of H, it is expedient to use the new variables defined in Chaps. 1 and 2, first, for example, to introduce the normal coordinates (1.3):

$$q_n = \sum_s e^{isn} \frac{Q_s}{N^{\frac{1}{2}}} \qquad p_n = \sum_s e^{-isn} \frac{P_s}{N^{\frac{1}{2}}} \tag{3.3}$$

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[1] See, e.g., L. I. Schiff, "Quantum Mechanics," 2d ed., p. 135, McGraw-Hill Book Company, Inc., New York, 1955.

In terms of these variables the commutation rules deduced by means of (1.5) become

$$\begin{aligned}[][Q_l,P_m] &= i\delta_{l,m}\\ [Q_l,Q_m] &= [P_l,P_m] = 0\end{aligned} \tag{3.4}$$

Since q_n and p_n are hermitian operators, $q_n^\dagger = q_n$, the conditions (1.4) have to be generalized to

$$Q_{-s} = Q_s^\dagger \qquad P_{-s} = P_s^\dagger \tag{3.5}$$

Inserting the new coordinates into the Hamiltonian, we find

$$\begin{aligned}H &= \tfrac{1}{2}\sum_s (P_s P_s^\dagger + \omega_s^2 Q_s Q_s^\dagger)\\ \omega_s^2 &= \Omega^2\left(2\sin\frac{s}{2}\right)^2 + \Omega_0^2\end{aligned} \tag{3.6}$$

In terms of the normal coordinates the oscillators are decoupled, and in accordance with (3.5) we now introduce, as in Chap. 2, the variables

$$\begin{aligned}a_s &= \frac{1}{(2\omega_s)^{\frac12}}(\omega_s Q_s + iP_s^\dagger)\\ a_s^\dagger &= \frac{1}{(2\omega_s)^{\frac12}}(\omega_s Q_s^\dagger - iP_s)\end{aligned} \tag{3.7}$$

or

$$\begin{aligned}Q_s &= \frac{1}{(2\omega_s)^{\frac12}}(a_s + a_{-s}^\dagger)\\ P_s &= -i\left(\frac{\omega}{2}\right)^{\frac12}(a_{-s} - a_s^\dagger)\end{aligned} \tag{3.8}$$

Note that $a_{-s} \neq a_s^\dagger$ and that we again have $2N$ independent operators a_s, $a_s^\dagger$ with $s = 2\pi l/N$ and $-N/2 \leq l \leq N/2$. The commutation relations for a_s and $a_s^\dagger$ follow directly from (3.4) and (3.7):

$$\begin{aligned}[][a_s,a_{s'}^\dagger] &= \delta_{s,s'}\\ [a_s,a_{s'}] &= [a_s^\dagger,a_{s'}^\dagger] = 0\end{aligned} \tag{3.9}$$

The Hamiltonian becomes a sum of terms of the form (2.6),

$$\begin{aligned}H &= \sum_s \tfrac{1}{2}\omega_s(a_s^\dagger a_s + a_{-s}^\dagger a_{-s} + 1)\\ &= \sum_s \omega_s(a_s^\dagger a_s + \tfrac{1}{2})\end{aligned} \tag{3.10}$$

and we may draw conclusions about its eigenvalues and eigenvectors as in the previous section.

The state of lowest energy $|\,0\rangle$ is determined by the condition

$$a_s\,|\,0\rangle = 0 \tag{3.11}$$

for all s. It belongs to the eigenvalue

$$E_0 = \sum_s \tfrac{1}{2}\omega_s \tag{3.12}$$

The general eigenvalue is given by

$$E = E_0 + \sum_s n_s \omega_s \tag{3.13}$$

where the n_s are a set of N nonnegative integers. The eigenvector belonging to the eigenvalue (3.13) is a generalization of (2.11):

$$| n_1, n_2, n_3, \ldots, n_N \rangle = (n_1!\, n_2!\, n_3! \cdots n_N!)^{-\frac{1}{2}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} \cdots (a_N^\dagger)^{n_N} | 0 \rangle \tag{3.14}$$

The fact that the eigenvalues of the energy are integer multiples of basic frequencies lends itself to a particle interpretation. The state (3.14) behaves like one with n_1 particles of energy ω_1, n_2 particles of energy ω_2, etc. Later, when we consider localized quantities, such as the energy or momentum contained in a certain volume, it will become apparent that the particle properties of the system are actually much more extensive than they now appear. Since our system just represents elastic (sound) waves, the quanta are usually called phonons. The energy of the quanta is additive, so that they behave like noninteracting particles. Furthermore, a state is characterized only by the number of particles in the N modes with energies ω_s, and there is no possibility of distinguishing the various particles in the same mode. Every mode can accommodate an arbitrary number of particles. Hence the particles obey Bose-Einstein statistics, and we have a model for particles which are indistinguishable. It appears that particles are more like vibrations than like classical bodies, and any two vibrations cannot differ so long as they have the same frequency. That particles lose their identity is one of the most revolutionary consequences of the application of quantum theory to fields. We shall not belabor the point, since it is discussed in most books on elementary wave mechanics. The correspondence between the elementary excitations of an elastic body and an ideal Bose gas forms the basis of the theory of specific heat of solids at low temperatures. Usually it is taken as granted that to any motion with a frequency ω there belongs an energy $\hbar\omega$. As we have seen, some mathematical development is necessary to deduce this result from first principles. Our idealization of purely harmonic forces is, of course, not always close to reality, but there are systems where the essential features of the Bose gas show up.

3.2. Quantum Features. It remains for us to study how the typical quantum features of the oscillator, such as the zero-point motion and energy, manifest themselves in our vibrating string. From (3.12) and (3.6), we see that the zero-point energy lies between $N\Omega_0/2$ and

$N(\Omega^2 + \Omega_0^2)^{\frac{1}{2}}/2$; it is thus the same as that of N uncoupled oscillators with basic frequencies lying between $N\Omega_0/2$ and $N(\Omega^2 + \Omega_0^2)^{\frac{1}{2}}/2$. In a crystal lattice this zero-point energy plays an important role, but for the fields of elementary particles it has not yet been possible to relate it to observable effects. In the latter case, since the number of degrees of freedom N goes to infinity, it becomes infinite. Relativistic invariance requires that it should be zero, since the state with no particles should look the same to observers in different Lorentz frames; a nonvanishing energy-momentum vector spoils this property. It is conventionally removed by calling the operator $H - E_0$ the energy. But one day, perhaps, its deeper significance will be discovered.

For the zero-point oscillation of the nth atom in the ground state of the system, we find, since $\langle 0 \mid q_n \mid 0 \rangle = 0$,

$$\begin{aligned}(\Delta q_n)^2 &= \langle 0 \mid q_n^2 \mid 0 \rangle \\ &= \sum_{s,s'} \frac{e^{i(s-s')n}}{N} \langle 0 \mid Q_s Q_{s'}^{\dagger} \mid 0 \rangle = \frac{1}{N} \sum_s \frac{1}{2\omega_s} \qquad (3.15)\end{aligned}$$

That is to say, the square fluctuation is just the mean value of the fluctuations associated with the frequencies ω_s. As was to be expected, the quantum fluctuations of the various modes are independent, so that the square fluctuations are additive. For atoms in a lattice the fluctuations have an amplitude which is somewhere between nuclear and atomic dimensions, and they are directly observable, for example, by the scattering of light or neutrons. The lighter the atoms and the weaker the forces between them the more violent are the fluctuations. As mentioned, they lead to macroscopic effects for helium, which they prevent from solidifying under normal pressure even at 0° temperature. Although this fluctuation is familiar from elementary quantum mechanics, the analogous result for fields is somewhat surprising and was only discovered in the modern development of quantum electrodynamics. We shall take this up in detail in a later chapter, where we shall study the fluctuation effects for states in which particles are present.

3.3. Dynamical Aspects. For the time dependence of the field operators we obtain, in analogy with (2.20) to (2.22),

$$\dot{a}_s(t) = -i\omega_s a_s(t) \qquad (3.16)$$

and

$$q_n(t) = \sum_s \left(\frac{1}{2N\omega_s}\right)^{\frac{1}{2}} [e^{i(sn-\omega_s t)} a_s(0) + e^{-i(sn-\omega_s t)} a_s^{\dagger}(0)] \qquad (3.17)$$

This form is identical with the time dependence of the classical solution of (1.8). It is the most general superposition of vibrations with eigenfrequencies ω_s, with the important difference that the coefficients are operators.

In analogy with the end of the last chapter, we can construct a standard wave packet in which the nth atom is, at the time $t = 0$, displaced from its equilibrium position by $d_{n'}$. A general state $| d_i\rangle$ of this kind is defined by

$$a_s \,| d_i\rangle = \left(\frac{\omega_s}{2N}\right)^{\frac{1}{2}} \sum_{n'} e^{-isn'} d_{n'} \,| d_i\rangle \tag{3.18}$$

for all s. For real d_n the expectation values of the positions at a time t correspond to the classical motion caused by an initial displacement d_n of the atoms and zero initial velocity:

$$\langle d \,| q_n(t) |\, d\rangle = \sum_{s,n'} \frac{1}{N} d_{n'} \cos\,[s(n - n') - \omega_s t] \tag{3.19}$$

A macroscopic sound wave with a single frequency ω_s and amplitude d can be represented by (3.18), with the d_n assuming the complex values $d_n = de^{is'n}$. Calling this state $| \,\omega_{s'}\rangle$, we have

$$a_s \,| \,\omega_{s'}\rangle = d\delta_{ss'} \left(\frac{N\omega_s}{2}\right)^{\frac{1}{2}} | \,\omega_{s'}\rangle$$

and hence it corresponds to a Poisson distribution of phonons with the appropriate frequency and a mean number of phonons $d^2\omega_{s'}/2$. For the time-dependent solution we find

$$\langle q_n(t) \,| \,\omega_{s'}\rangle = \psi_{d_s'}[q_n(t)] \propto \exp\left[-\frac{\omega_{s'}}{2}\Big(q_n - \sum_{n'} d_{n'} e^{i(s'n - s'n' - \omega_{s'} t)}\Big)^2\right]$$

and the average positions are in this case simply given by

$$\langle d \,| q_n |\, d\rangle = d \cos\,(s'n - \omega_{s'} t) \tag{3.20}$$

If we want this classical-like motion to be observable, it is necessary that the displacement d be much larger than the zero-point fluctuation amplitude. The latter can be seen from (3.15) to be of the order of $1/(\omega_{s'})^{\frac{1}{2}}$ if all frequencies are of the order of $\omega_{s'}$. Since the mean number of phonons is $d^2\omega_{s'}/2$, the above condition implies that it is $\gg 1$. In a state with a definite number n' of phonons, the expectation value of all q_n (e.g., $\langle n' \,| q_n |\, n'\rangle$) is zero. This is usually expressed by saying that the phases of the waves associated with phonons are completely undetermined.

In summary, we can say that sound waves and phonons represent the classical and quantum-mechanical aspects of vibrating systems and are generalizations of what we found for the harmonic oscillator.

CHAPTER 4

Fields

4.1. Continuously Coupled Oscillators. We shall now investigate how the quantum theoretic development works in the limit of a continuous line. In the limit $N \to \infty$, $a \to 0$, but aN finite (see Chap. 1),

$$q_n \to q(x)a^{\frac{1}{2}} \qquad \sum_n \to a^{-1}\int dx \qquad \Omega \to \frac{v}{a}$$

the Hamiltonian and the field equations (3.1) become

$$\begin{gathered} H = \int_0^L dx\, \tfrac{1}{2}\left[p^2(x,t) + v^2\left(\frac{\partial q}{\partial x}\right)^2 + \Omega_0^2 q^2(x,t)\right] \\ \ddot{q}(x,t) - v^2 \frac{\partial^2}{\partial x^2} q(x,t) + \Omega_0^2 q^2(x,t) = 0 \end{gathered} \tag{4.1}$$

We have already remarked, in Chap. 1, that even in the continuum limit the normal coordinates form an enumerable set. Therefore we shall first quantize the theory in terms of these variables and shall study the commutation rules for the continuous coordinates $q(x)$ later on. Rewriting (3.3) in the form (1.10), with $s \to ka$,

$$\begin{gathered} q(x) = L^{-\frac{1}{2}} \sum_k e^{ikx} Q_k \qquad p(x) = L^{-\frac{1}{2}} \sum_k e^{-ikx} P_k \\ k = \frac{2\pi l}{L} \qquad -\infty < l < \infty \end{gathered} \tag{4.2}$$

we find for the energy, as in (1.12),

$$\begin{gathered} H = \sum_k \tfrac{1}{2}(P_k P_k^\dagger + \omega_k^2 Q_k Q_k^\dagger) \\ \omega_k^2 = v^2 k^2 + \Omega_0^2 \end{gathered} \tag{4.3}$$

where we have used the continuum limit of (1.5),

$$\int_0^L dx\, e^{i(k-k')x} = L\delta_{k\,k'}$$

This is now a sum over an infinite number of uncoupled oscillators. In terms of the labels k, the commutation rules (3.2) are

$$\begin{aligned}[][Q_k,P_{k'}] &= i\delta_{k,k'} \\ [Q_k,Q_{k'}] &= [P_k,P_{k'}] = 0\end{aligned} \tag{4.4}$$

and similarly those for the operators a and $a^\dagger$ of (3.7) are

$$\begin{aligned}[][a_k,a_{k'}^\dagger] &= \delta_{k,k'} \\ [a_k,a_{k'}] &= [a_k^\dagger,a_{k'}^\dagger] = 0\end{aligned}$$

The introduction of these operators allows us to write the following expression for the energy:

$$H = \sum_k \omega_k(a_k^\dagger a_k + \tfrac{1}{2}) \tag{4.5}$$

In correspondence to the development of Chap. 3, we find that the eigenvalues of $H - E_0$ are integer multiples of the ω_k. We shall see in the next chapters that k has the significance of the momentum of the particles with energy ω_k. Hence, if v is the velocity of light and $\Omega_0 = mc^2$, then the energy and momentum of a field quantum are related in the same manner as those of a relativistic particle.

In the continuum form it is easy to generalize to the three-dimensional case. In the mechanical model of a displacement field which we had in mind so far, the general three-dimensional case is somewhat more complicated, since the field is then a vector and has three components. However, by only allowing displacements of a three-dimensional atomic lattice in one direction, say x, as shown in Fig. 4.1, we have the discrete analogue of a scalar (hermitian) field $\phi(x,y,z)$. It is this simpler case which will prove to be an appropriate description for pions if we identify v with the velocity of light and Ω_0 with the mass of the pion. The three-dimensional version of (3.1) is[1]

$$\begin{gathered} H = \tfrac{1}{2}\int d^3r\,\{\dot\phi(\mathbf{r},t)^2 + [\nabla\phi(\mathbf{r},t)]^2 + m^2\phi^2(\mathbf{r},t)\} \\ \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(\mathbf{r},t) = 0 \end{gathered} \tag{4.6}$$

which is the Klein-Gordon equation.

Anticipating future notation, we have put $v = c$ equal to 1 and replaced Ω_0 by m. The three-dimensional periodicity condition requires

---

[1] We shall use $\mathbf{r}$ as an abbreviation for the three space coordinates (x_1,x_2,x_3) or x, y, z; $(\mathbf{r},t)$ for (x,y,z,t); and r for $|\mathbf{r}|$ (e.g., $\mathbf{r}^2 = r^2$). Frequently we shall write $(\mathbf{r},0)$ simply as $\mathbf{r}$.

$$\phi(x+L,y,z,t) = \phi(x,y+L,z,t) = \phi(x,y,z+L,t) = \phi(x,y,z,t) \quad (4.7)$$

These conditions and the equations of motion (4.6) can be satisfied in analogy to (3.17) by[1]

$$\phi(\mathbf{r},t) = \frac{1}{L^{\frac{3}{2}}} \sum_{\mathbf{k}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}a_{\mathbf{k}} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}a_{\mathbf{k}}^{\dagger}}{(2\omega_k)^{\frac{1}{2}}}$$
$$k_{x,y,z} = \frac{2\pi l_{x,y,z}}{L} \quad (4.8)$$

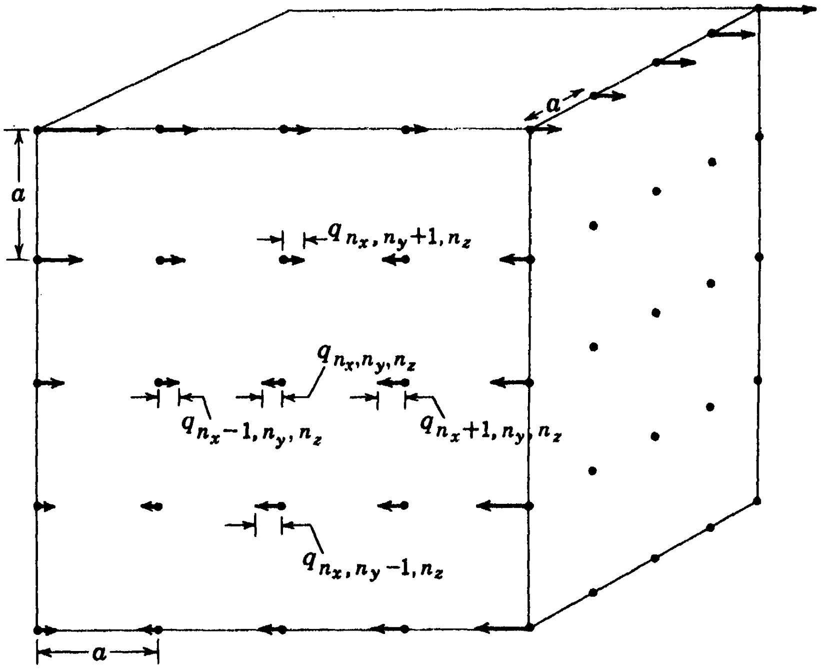

Fig. 4.1. Mechanical analogue of a scalar field. The cube of vibrating atoms, of length L, has an atomic equilibrium separation of a. All atoms vibrate in a single direction, here chosen to be x.

The commutation rules (3.9) are generalized to

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k},\mathbf{k}'}$$
$$[a_{\mathbf{k}},a_{\mathbf{k}'}] = [a_{\mathbf{k}}^{\dagger},a_{\mathbf{k}'}^{\dagger}] = 0 \quad (4.9)$$

We shall shortly develop a more general recipe for finding the commutation properties of the field. The Hamiltonian can be written in the familiar form (3.10), except that the sum over **k** is now a three-dimensional one:

$$H = \sum_{\mathbf{k}} \omega(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + \tfrac{1}{2}) \quad (4.10)$$

---

[1] We shall henceforth abbreviate ω_k by ω, $\omega_{k'}$ by ω', etc.

Thus we get the remarkable result that, by applying the rules of quantum mechanics to a field which obeys the Klein-Gordon equation, we obtain a system that behaves like an ensemble of an unlimited number of relativistic Bose particles. To be more specific, we have, by analogy with (3.14) and (3.13), a state

$$\begin{aligned} | n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}, \ldots \rangle &\equiv | n_1, n_2, n_3, \ldots \rangle \\ &= \frac{1}{(n_1!\, n_2!\, n_3! \cdots)^{\frac{1}{2}}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} \cdots | 0,0,0, \ldots \rangle \end{aligned}$$

which is an eigenstate of the Hamiltonian H with energy E,¶

$$\begin{aligned} H \,| n_1, n_2, \ldots \rangle &= \sum_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \omega + \tfrac{1}{2}\omega) \,| n_1, n_2, \ldots \rangle \\ &= E \,| n_1, n_2, \ldots \rangle = \sum_i \,(n_i \omega_i + \tfrac{1}{2}\omega_i) \,| n_1, n_2, \ldots \rangle \end{aligned}$$

The eigenfunctions therefore satisfy a Schrödinger equation for an unlimited number of particles with energies given by

$$\omega_i = (k_i^2 + m^2)^{\frac{1}{2}}$$

Since the application of $a^\dagger$ to a ket with n particles yields one with $n + 1$ particles, $a^\dagger$ is usually called a creation operator and a, correspondingly, a destruction operator.

There are other fields for which the Hamiltonian is not the continuum analogue of coupled oscillators but the Larmor precession of electron spins. In this case one finds that quantization leads to particles obeying Fermi-Dirac statistics. The kinds of fields that correspond to particles with half-odd-integral spin are beyond the scope of this book. However, we should like to point out that quantum field theory predicts the experimentally established connection between spin and statistics.[1]

Roughly speaking, going to three dimensions increases the number of degrees of freedom by a factor of 3. This change is not so drastic as that of the limiting procedure $N \to \infty$ used in this chapter.

Finally, we shall chiefly discuss the limit $L \to \infty$, where the unphysical boundary condition (4.7) is relaxed. In this limit the **k** vectors in (4.8) become a dense set such that the k_i are now a continuous variable going from $-\infty$ to ∞. We shall emphasize this by denoting the destruction and creation operators in this limit by $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$. Furthermore $\sum_{\mathbf{k}}$ then has to be replaced by an integral. Since the distance between two neighboring points in $\mathbf{k}$ space is $2\pi/L$, their density is $(L/2\pi)^3$, and hence in the limit

---

¶ This statement and the equation that follows can also be proved directly, by means of the communication relations (4.9).

[1] W. Pauli, in W. Pauli (ed.), "Niels Bohr and the Development of Physics," p. 30, McGraw-Hill Book Company, Inc., New York, 1955.

$$\mathop{\mathsf{S}}\limits_{\mathbf{k}} \equiv L^{-3}\sum_{\mathbf{k}} \rightarrow (2\pi)^{-3}\int d^3k$$

In most calculations leading to numerical results, this suitably weighted sum over all degrees of freedom occurs, and hence we introduce a new notation $\mathop{\mathsf{S}}\limits_{\mathbf{k}}$ for it.

This introduction of infinitely many degrees of freedom raises some questions, which we shall now discuss. First of all, since there are infinitely many ω and they do not have an upper bound, the zero-point energy

$$E_0 = \tfrac{1}{2}\sum_{\mathbf{k}} \omega \tag{4.11}$$

diverges. This can be remedied by calling

$$H = \sum_{\mathbf{k}} \omega a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \tag{4.12}$$

the energy, since at this moment we do not know what the zero-point energy of the field is. One could ask whether the infinite sum (4.12) converges toward a limit. This raises difficult questions about non-separable Hilbert spaces which we are not prepared to answer. The reader has to be content with the observation that for states with only a finite number of excited oscillators the application of H gives a finite sum.

For the other typical quantum feature, namely, the zero-point fluctuations of $\phi(\mathbf{r})$, the infinite number of degrees of freedom also creates some difficulties. As in (2.15) and (3.15), we find

$$\langle 0 \mid \phi(\mathbf{r}) \mid 0 \rangle = 0$$

$$[\Delta\phi(\mathbf{r})]^2 = \langle 0 \mid \phi^2(\mathbf{r}) \mid 0 \rangle = \mathop{\mathsf{S}}\limits_{\mathbf{k}} \frac{1}{2\omega} = \mathop{\mathsf{S}}\limits_{\mathbf{k}} \frac{1}{2(\mathbf{k}^2 + m^2)^{1/2}} \tag{4.13}$$

which diverges. This can best be seen in the limit $L \rightarrow \infty$, where we obtain

$$[\Delta\phi(\mathbf{r})]^2 = (2\pi)^{-3}\int d^3k \, \frac{1}{2(\mathbf{k}^2 + m^2)^{1/2}} \rightarrow \infty \tag{4.14}$$

The infinite fluctuation is connected with the fact that $\phi(\mathbf{r})$ gives a state with an infinite norm when applied to any state with finite energy.[1] Thus $\phi(\mathbf{r})$ is not an operator in the Hilbert space we are dealing with. However, it turns out that the average of $\phi(\mathbf{r})$ over a finite region in space has a finite square fluctuation. To see this, we define the average field ϕ_b over a volume b^3 by

---

[1] Equation (4.14) is a special case of this statement for the vacuum state.

$$\bar{\phi}_b = (2\pi b^2)^{-\frac{3}{2}} \int d^3r\, e^{-\mathbf{r}^2/2b^2} \phi(\mathbf{r})$$

and obtain

$$\begin{aligned} \langle 0 \,|\, \bar{\phi}_b^2 \,|\, 0 \rangle &= (2\pi)^{-6} b^{-6} \int d^3k\, d^3r\, d^3r' \, \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{2\omega} \, e^{-(r^2+r'^2)/2b^2} \\ &= \frac{1}{(2\pi)^3} \frac{1}{2} \int \frac{d^3k}{2\omega} e^{-k^2 b^2} \approx \frac{b^{-3}}{(b^{-2}+m^2)^{\frac{1}{2}}} \end{aligned} \tag{4.15}$$

This teaches us that the fluctuations of the field become more and more violent as we decrease the volume b^3 over which we average. Since the averaging process renders wavelengths less than b ineffective, a decrease in the volume increases the contributions to the field fluctuations.

At first sight this seems to have drastic consequences. The electromagnetic potentials V and $\mathbf{A}$ satisfy an equation of the type (4.6) with $m = 0$. Therefore, for the fluctuation in V we obtain

$$\Delta\bar{V} \sim \frac{1}{b}$$

which is enormous if we keep in mind that in our units[1] the elementary charge $e = (4\pi/137)^{\frac{1}{2}}$. That is to say, the potential $e/4\pi b$ created by the elementary charge at a distance b is much less than the quantum fluctuations of the field averaged over a comparable region. One might wonder how, in these circumstances, the electron in a hydrogen atom can possibly follow the orbit dictated by the force of the proton. The answer is that most of the fluctuations have a frequency $\sim b^{-1} = me^2$, which for $b \sim 10^{-8}$ cm is 137 times the frequency of the electron in the ground state. They merely cause a small-amplitude, high-frequency vibration of the electron, whereas the Coulomb field acts for relatively long times in the same direction and dominates the motion. We easily[2] find that the amplitude of this vibration is less than the Compton wavelength of the electron, 10^{-11} cm. Therefore this effect displaces atomic levels less than relativistic effects, which spread the charge of the electron over a region of the size of the Compton wavelength. This will be shown for scalar field particles in Chap. 5. Nevertheless, the present experiments establish the influence of the quantum fluctuations in the hydrogen atom with an accuracy of 1 part in 10^4.

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[1] We remind the reader that $\hbar = c = 1$.

[2] See W. Thirring, "Principles of Quantum Electrodynamics," Academic Press, Inc., New York, 1958; and T. A. Welton, *Phys. Rev.*, **74**: 1157 (1948).

In the mesodynamic application which we shall discuss, the vacuum fluctuations of the field will be particularly important, because the meson-nucleon interaction is much stronger than e. The fluctuations constantly shake the spin and charge of the nucleon, since the pion field acts mainly as a torque on these variables rather than on the position of the nucleon to which it is coupled.

4.2. Derivation of Field Equations from a Lagrangian. We shall now study the form of the commutation rules for the continuous variables $\phi(r)$ and $\dot{\phi}(r)$. This will give us a clue to the general quantization rules. Using (4.8) and (4.9), we obtain

$$
\begin{aligned}
&[\phi(\mathbf{r},t),\ \phi(\mathbf{r}',t')]_{t=t'} = [\dot{\phi}(\mathbf{r},t),\ \dot{\phi}(\mathbf{r}',t')]_{t=t'} = 0 \\
&[\phi(\mathbf{r},t),\ \dot{\phi}(\mathbf{r}',t')]_{t=t'} = i\delta^3(\mathbf{r}-\mathbf{r}') \qquad (4.16)
\end{aligned}
$$

Here we have used a fact known from the theory of Fourier expansions, namely, that the following sum is effectively a δ function for $-L/2 < x < L/2$.¶ For the limit $L \to \infty$ we have

$$\mathop{\mathrm{S}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = \left(\frac{1}{2\pi}\right)^3 \int d^3k\, e^{i\mathbf{k}\cdot\mathbf{r}} = \delta^3(\mathbf{r}) \qquad (4.17)$$

This expresses the completeness of the exponential functions and is the continuum analogy of (1.5). That (4.16) is the continuum form of the canonical commutation rules was to be expected. In fact, the limit for $a \to 0$ of (3.2), $[q_l,p_m] = i\delta_{lm}$, is

$$[q(x), p(x')] = [q(x), \dot{q}(x')] = i\,\frac{\delta_{x,x'}}{a}$$

where $\delta_{x,x'}$ equals 1 if x and x' are in the same lattice space and equals 0 otherwise. For the limit $a \to 0$ the ratio $\delta_{x,x'}/a$ is just the one-dimensional Dirac δ function, $\delta(x - x')$, and (4.16) is the three-dimensional generalization of this form.

We can now state the general rules for quantizing a field with the aid of the formal tools of the functional derivative and the δ function. It is convenient to start with the Lagrangian, from which we get the field equations as the stationary properties of the action integral,

$$W = \int_{t_1}^{t_2} L(\phi,\nabla\phi,\dot{\phi})\, dt = \int_{t_1}^{t_2} dt\, d^3r\, \mathscr{L}(\phi,\nabla\phi,\dot{\phi})$$

where $\mathscr{L}$ is the Lagrangian density. With the boundary conditions that the arbitrary variations $\delta\phi$ be zero at t_1 and t_2 and by means of the functional derivative introduced in (1.13), we find

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¶ See, e.g., L. I. Schiff, "Quantum Mechanics," 2d ed., p. 52, McGraw-Hill Book Company, Inc., New York, 1955.

$$\begin{aligned}
\delta W &= \int_{t_1}^{t_2} dt \left(\frac{\delta L}{\delta \dot{\phi}}\, \delta\dot{\phi} + \frac{\delta L}{\delta \phi}\, \delta\phi \right) \\
&= \int_{t_1}^{t_2} dt \int d^3r \left[-\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} + \frac{\partial \mathscr{L}}{\partial \phi} + \frac{\partial \mathscr{L}}{\partial(\nabla\phi)}\,\nabla \right] \delta\phi \\
&= \int_{t_1}^{t_2} dt \int d^3r\, \delta\phi \left[-\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} + \frac{\partial \mathscr{L}}{\partial \phi} - \nabla\frac{\partial \mathscr{L}}{\partial(\nabla\phi)} \right]
\end{aligned}$$

The Euler equations of motion are therefore

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \frac{\partial \mathscr{L}}{\partial \phi} - \nabla\frac{\partial \mathscr{L}}{\partial(\nabla\phi)}$$

By means of the functional derivatives

$$\frac{\delta L}{\delta \dot{\phi}} = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \qquad \text{and} \qquad \frac{\delta L}{\delta \phi} = \frac{\partial \mathscr{L}}{\partial \phi} - \nabla\frac{\partial \mathscr{L}}{\partial(\nabla\phi)}$$

the Euler equations take on the classical form

$$\frac{\partial}{\partial t}\frac{\delta L}{\delta \dot{\phi}(\mathbf{r},t)} = \frac{\delta L}{\delta \phi(\mathbf{r},t)} \tag{4.18}$$

The generalization of the canonical conjugate variable is

$$\pi(\mathbf{r},t) = \frac{\delta L}{\delta \dot{\phi}(\mathbf{r},t)}$$

For the Hamiltonian and the commutation relations we postulate accordingly the general formulas

$$\begin{aligned}
H &= \int d^3r\, \pi(\mathbf{r},t)\dot{\phi}(\mathbf{r},t) - L \\
[\phi(\mathbf{r},t), \pi(\mathbf{r}',t')]_{t=t'} &= i\delta^3(\mathbf{r}-\mathbf{r}') \qquad\qquad (4.19) \\
[\phi(\mathbf{r},t), \phi(\mathbf{r}',t')]_{t=t'} &= [\pi(\mathbf{r},t), \pi(\mathbf{r}',t')]_{t=t'} = 0
\end{aligned}$$

Hence the transition from discrete to continuous variables is simply done by replacing sum by integral, partial derivative by functional derivative, and Kronecker δ by Dirac δ.

The field equations and Hamiltonian (4.6) are derived from the Lagrangian[1]

$$L(t) = \int d^3r\, \tfrac{1}{2}[\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2] \tag{4.20}$$

With the aid of (1.13), the canonical conjugate field $\pi(\mathbf{r},t)$ is seen to be

$$\pi(\mathbf{r},t) = \dot{\phi}(\mathbf{r},t) = \frac{\delta L}{\delta\dot{\phi}(\mathbf{r},t)}$$

and hence this prescription leads to the form of the Hamiltonian (4.6) and commutation rules (4.4) and (4.16).

The use of a mechanical analogy to find the field equations (4.6) may not seem very convincing when applied to, say, the pion field. To do this more systematically, the Lagrangian formalism is essential. To satisfy Lorentz invariance, the Lagrangian density, e.g., the quantity under the integral in (4.20), has to be a scalar. Our expression is, in fact, the most general scalar which is quadratic in the field and its derivatives.

As a further example, which we shall occasionally use to contrast with the relativistic field ϕ, we apply the Lagrangian formalism to a field ψ which obeys the time-dependent Schrödinger equation. This equation is of the first order in the time, but since the field is not hermitian, $\psi^\dagger \neq \psi$, the two equations for ψ and $\psi^\dagger$ are equivalent to one equation of second order. The application of our rules to equations of first order requires some care. To see this, we revert temporarily to the study of a single harmonic oscillator. The equation of motion with $\omega = 1$ for the real operator q, $\ddot{q} = -q$, can be rewritten in terms of two first-order differential equations for the nonhermitian operators $\mathscr{Q}$ and $\dot{\mathscr{Q}}$:

$$\begin{aligned} \mathscr{Q} &= q - i\dot{q} \qquad & \dot{\mathscr{Q}} &= i\mathscr{Q} \\ \mathscr{Q}^\dagger &= q + i\dot{q} \qquad & \dot{\mathscr{Q}}^\dagger &= -i\mathscr{Q}^\dagger \end{aligned}$$

We recognize that a hermitian Lagrangian which gives $\dot{\mathscr{Q}} = i\mathscr{Q}$ and the usual energy is

$$L = -\frac{i}{4}(\mathscr{Q}^\dagger\dot{\mathscr{Q}} - \dot{\mathscr{Q}}^\dagger\mathscr{Q}) - \tfrac{1}{2}\mathscr{Q}^\dagger\mathscr{Q}$$

The conjugate variables π and $\pi^\dagger$ are

$$\pi = \frac{\delta L}{\delta\dot{\mathscr{Q}}} = -\frac{i\mathscr{Q}^\dagger}{4} \qquad \pi^\dagger = \frac{\delta L}{\delta\dot{\mathscr{Q}}^\dagger} = \frac{i\mathscr{Q}}{4}$$

and hence the Hamiltonian is[2]

$$H = \pi\dot{\mathscr{Q}} + \pi^\dagger\mathscr{Q}^\dagger - L = \tfrac{1}{2}\mathscr{Q}^\dagger\mathscr{Q} = \tfrac{1}{2}(q^2 + \dot{q}^2) + \tfrac{1}{2}$$

---

[1] This Lagrangian is not unique, since it can be changed by adding a total time derivative.

[2] $\mathscr{Q}$ and $\mathscr{Q}^\dagger$ are independent variables.

However, the correct commutation rules derived from $[q,\dot{q}] = i$ are[1]

$$[\mathscr{Q}^\dagger,\mathscr{Q}] = 2 \qquad [\mathscr{Q},\dot{\mathscr{Q}}] = [\mathscr{Q}^\dagger,\dot{\mathscr{Q}}^\dagger] = 0$$

and imply

$$[\mathscr{Q},\pi] = \frac{i}{2} \qquad [\mathscr{Q}^\dagger,\pi^\dagger] = \frac{i}{2}$$

The factor of $\frac{1}{2}$ which appears in the commutation relations for the canonically conjugate operators $\mathscr{Q}$ and π and $\mathscr{Q}^\dagger$ and $\pi^\dagger$ will always[2] be present if a hermitian Lagrangian is used to derive first-order equations of motion for a nonhermitian field.

A suitable hermitian Lagrangian for the Schrödinger fields ψ and $\psi^\dagger$ is

$$L = \int d^3r\, \tfrac{1}{2}\Big[i(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) - \frac{1}{m}\nabla\psi^\dagger \cdot \nabla\psi\Big] \tag{4.21}$$

and the field equations derived from it are[3]

$$i\dot{\psi} = -\frac{\nabla^2\psi}{2m}$$

$$i\dot{\psi}^\dagger = \frac{\nabla^2\psi}{2m}$$

The momentum operators π, $\pi^\dagger$ canonically conjugate to ψ and $\psi^\dagger$ are

$$\pi = \frac{i}{2}\psi^\dagger \qquad \pi^\dagger = -\frac{i}{2}\psi$$

By means of the commutation rules

$$\begin{aligned}
&[\psi(\mathbf{r}), \pi(\mathbf{r})] = [\psi^\dagger(\mathbf{r}), \pi^\dagger(\mathbf{r})] = \frac{i}{2}\,\delta^3(\mathbf{r} - \mathbf{r}') \\
&[\psi(\mathbf{r}), \pi^\dagger(\mathbf{r})] = [\psi^\dagger(\mathbf{r}), \pi(\mathbf{r})] = 0
\end{aligned} \tag{4.22}$$

we therefore find

$$[\psi(\mathbf{r},t), \psi^\dagger(\mathbf{r}',t')]_{t=t'} = \delta^3(\mathbf{r} - \mathbf{r}')$$

$$[\psi(\mathbf{r},t), \psi(\mathbf{r}',t')]_{t=t'} = [\psi^\dagger(\mathbf{r},t), \psi^\dagger(\mathbf{r}',t')]_{t=t'} = 0$$

---

[1] We have chosen a hermitian Lagrangian because it is identical with the conventional one for a harmonic oscillator. By adding a term $i(d/dt)(\mathscr{Q}^\dagger\mathscr{Q})$, we can start from a nonhermitian Lagrangian which leads to the correct equations of motion but to the commutation rules $[\mathscr{Q}^\dagger,\mathscr{Q}] = 2$, $[\mathscr{Q},\pi] = 0$, $[\mathscr{Q}^\dagger,\pi^\dagger] = i$. See, e.g., Schiff, *op. cit.*, p. 348.

[2] This applies, of course, only to the type of system under consideration, namely, one described by linear differential equations, which are of first order in time.

[3] This case is actually a limiting case of the Klein-Gordon field,

$$\psi(\mathbf{r},t) = \lim_{m\to\infty} e^{imt}\phi(\mathbf{r},t)$$

Finally, the Hamiltonian is

$$H = \int d^3r \, \frac{1}{2m} \nabla\psi^\dagger \cdot \nabla\psi = \int d^3r \, \frac{i}{2m} (\nabla\psi^\dagger \cdot \nabla\pi^\dagger - \nabla\pi \cdot \nabla\psi)$$

Care must be exercised in deriving the field equations from the Hamiltonian. It is only when the canonically conjugate momenta π, $\pi^\dagger$ appear explicitly in a hermitian Hamiltonian that the relations (1.14) give the correct field equations. The eigenvalues of the energy can be obtained by the same manipulations as before:[1]

$$\psi(\mathbf{r},t) = L^{-\frac{3}{2}} \sum_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - E_k t)} a_{\mathbf{k}} \qquad E_k = \frac{k^2}{2m}$$

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0 \tag{4.23}$$

$$H = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} E_k$$

The Schrödinger field case is actually somewhat simpler than the relativistic Klein-Gordon one; in particular, the relation between the energy and momentum of the field quanta is the classical one. In the next chapter we shall study some differences between the relativistic and the nonrelativistic case which are not of a trivial kinematical nature.

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[1] Since ψ is not hermitian, the creation part with $a_{\mathbf{k}}^\dagger$ is not needed.

CHAPTER 5

Observables

5.1. Energy, Momentum, and Angular Momentum. Led by our mechanical analogue we have so far investigated only two observables: the total energy and the field amplitude $\phi(\mathbf{r},t)$. In the continuum limit the latter was not an operator in the sense that, when applied to a state of finite norm, it leads to a state of infinite norm [see (4.14)], so that we shall need some other observables for a discussion of the physical properties of our quantized field. There are some general recipes in classical field theory for constructing quantities such as the linear momentum or the angular momentum of a field from a given Lagrangian. These and other observables, together with their commutation properties, will be studied in this chapter, and the next one will be devoted to the eigenstates of these operators.

As in point mechanics, the invariance of the Lagrangian under certain transformations ensures the existence of corresponding constants of the motion. We have already encountered one example of this general principle, namely, the energy. If, and only if, the Lagrangian does not depend explicitly on time, then the energy (4.19) is constant. The reader will readily verify the formula

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{5.1}$$

where $\partial L/\partial t$ does not involve the implicit time dependence through $\phi(\mathbf{r},t)$. Similarly, if L does not depend explicitly on the coordinates $\mathbf{r}$ (i.e., those Lagrangians studied in Chap. 4), which means that it is invariant under displacements and rotations in space, then we get six more constants of the motion, one for each parameter of the invariance group. In classical mechanics the constants (which for displacements and rotations are the total momentum and angular momentum, respectively) associated with invariances are simply the generators of the transformation. The invariance of the classical Hamiltonian under such transformations ensures the vanishing of the Poisson bracket between the Hamiltonian and the generators, which implies that the latter are constant. The same holds true in quantum theory, where the Poisson bracket is replaced by the commutator. Therefore, the linear and angular momenta are generally defined to be those operators for which the commutator with any quantity gives its change under an infinitesimal displacement and rotation.

At this point we have to remember that our problem is not yet invariant under rotations, because of the cubic periodicity condition (4.8) we imposed on our fields. The invariance is obtained, however, by imposing a spherical boundary condition, e.g.,

$$\phi(\mathbf{r},t) = 0 \qquad \text{for } r = R \tag{5.2}$$

We shall have this condition in mind when discussing the total angular momentum. It is shown in the classical study of solids that the particular form of the boundary condition is unimportant for large systems and only serves as an aid for the mathematical development. The case of physical interest is the one with $L \to \infty$ or $R \to \infty$. Correspondingly, the form of the boundary condition should not, and does not, enter into results of physical significance which correspond to volume and not to surface effects. The physical results will always be deduced with states wherein the field is only excited in finite regions of space. For these states the field operators at infinity are effectively zero. With this in mind, we shall henceforth also neglect surface integrals from infinitely remote surfaces.

For the relativistic and nonrelativistic fields, the total momentum and angular momentum turn out to be[1]

$$\begin{aligned}
\mathbf{P} &= -\tfrac{1}{2}\int d^3r\,[\pi\nabla\phi + (\nabla\phi)\pi] \\
\mathbf{P} &= -\tfrac{1}{2}\int d^3r\,[\pi\nabla\psi + (\nabla\psi^\dagger)\pi^\dagger] \\
\mathbf{L} &= -\tfrac{1}{2}\int d^3r\,[\pi\mathbf{r}\times\nabla\phi + \mathbf{r}\times(\nabla\phi)\pi] \\
\mathbf{L} &= -\tfrac{1}{2}\int d^3r\,[\pi\mathbf{r}\times\nabla\psi + \mathbf{r}\times(\nabla\psi^\dagger)\pi^\dagger]
\end{aligned} \tag{5.3}$$

---

[1] These operators are restricted by conditions of hermiticity and proper behavior under Lorentz transformations. They can be obtained by analogy with classical mechanics. See, e.g., G. Wentzel, "The Quantum Theory of Fields," p. 8 and Appendix I, Interscience Publishers, Inc., New York, 1949. Here we shall merely give P and L and show that they have the correct properties associated with such operators.

With the aid of (4.19) we find, e.g.,

$$[\mathbf{P}, \phi(\mathbf{r},t)] = i\nabla\phi(\mathbf{r},t)$$
$$[\mathbf{L}, \phi(\mathbf{r},t)] = i\mathbf{r} \times \nabla\phi(\mathbf{r},t) \tag{5.4}$$

and similar equations for $\pi(\mathbf{r},t)$, ψ, etc. That is to say, the commutator of **P** and **L** with a field operator gives the change of that quantity under an infinitesimal displacement and rotation, respectively. Since the Hamiltonian is invariant under these operations, we have

$$[\mathbf{P},H] = [\mathbf{L},H] = 0 \tag{5.5}$$

which, because of (2.19), means that **P** and **L** are constant in time. In fact, (5.5) can also be verified with the aid of the field equations. We readily see that the expressions for $\dot{\mathbf{P}}$ and $\dot{\mathbf{L}}$ can be converted to infinite surface integrals by means of the Klein-Gordon equation. However, a simple calculation shows that **P** and **L** fail to commute; in fact, the commutation relations between them are the same as in elementary quantum mechanics,

$$[P_i,L_j] = i\epsilon_{ijk}P_k \qquad i, j, k = 1, 2, 3, \text{ or } x, y, z \tag{5.6}$$

where ϵ_{ijk} is the totally antisymmetric tensor of third rank, ϵ_{123} being 1 and ϵ_{213} being -1, for example. There is actually a very general reason for (5.6). Since **L** and **P** generate infinitesimal rotations and displacements, the commutation relation between them must be the same as the one for the operations of rotation and displacement. Similarly, the commutation rules of the components of L are worked out to be of the usual form

$$[L_i,L_j] = i\epsilon_{ijk}L_k \tag{5.7}$$

Inserting the expressions (4.8) or (4.23) into (5.3), we obtain, by our usual methods,[1]

$$\mathbf{P} = \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}\mathbf{k} \tag{5.8}$$

As we found earlier, the operators $a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$ have integer eigenvalues, and this tells us that the state

$$|\, n_1,n_2,\ldots\rangle \equiv |\, n_{\mathbf{k}_1},n_{\mathbf{k}_2},\ldots\rangle = (a_1^{\dagger})^{n_1}(a_2^{\dagger})^{n_2}\cdots|\,0\rangle$$

is also an eigenstate of **P** and belongs to the eigenvalue $n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + \cdots$. Our particle interpretation is thus supported by (5.8), which states that a momentum **k** is associated with the energy $\omega = (k^2 + m^2)^{\frac{1}{2}}$.¶ The eigenvalues for the momentum are therefore the integer multiples of the **k**. This resemblance to the energy eigenvalue problem has its formal origin in the fact that the commutation relations (4.19) and (5.4) have the same structure. Consequently, the possible values for the angular momentum are also of the same nature. However, our standard states, which are eigenstates of **P**, will not be eigenstates of **L**, since **P** and **L** do not commute except for eigenstates with $\mathbf{P} = 0$. But **L** and H commute, so that we should be able to find simultaneous eigenstates of, say, L_3 and H.

---

[1] The zero-point momentum $\frac{1}{2}\sum_{\mathbf{k}} \mathbf{k}$, which appears for the Klein-Gordon field, is zero by symmetry, because for every component k_i, there is one $-k_i$.

¶ Because of this, the $a_{\mathbf{k}}$ are usually called the particle-destruction operators in momentum (or **k**) space, as opposed to the $\phi(\mathbf{r})$ in coordinate (or **r**) space, which both create and destroy particles.

To construct such states, we should not expand in terms of plane waves (eigenfunctions of displacement), but rather in terms of spherical harmonics (eigenfunctions of rotations), since it is in the latter representation that we expect L_3 to be diagonal. To accomplish this objective, we use the plane-wave expansion

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{k}\sum_{l,m} (2\pi)^{\frac{3}{2}} i^l U_k^l(r) Y_l^{m*}(\theta_k,\varphi_k) Y_l^m(\theta_r,\varphi_r)$$

where θ_k, φ_k and θ_r, φ_r are the angles between the vectors **k** and **r** and an arbitrary z axis and where Y_l^m is a normalized spherical harmonic. The functions $U_k^l(r)$ satisfy the equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2\right] U_k^l(r) = 0 \tag{5.9}$$

and are given by

$$U_k^l(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} k j_l(kr)$$

The expansion (5.9) reduces to the familiar expression[1]

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta)$$

when **r** is taken to be in the direction of the z axis. However, since it is a rotationally invariant expression, it must hold generally.

The constants which appear in the definition of U_k^l have been chosen such that these functions have δ-function normalization. This can be seen by the use of their asymptotic behavior

$$U_k^l(r) = \frac{v_k^l(r)}{r} \qquad \lim_{r\to\infty} v_k^l(r) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin\left(kr - \frac{l\pi}{2}\right)$$

---

[1] This expansion is most easily obtained by comparing the asymptotic expansions of both sides after integrating with $P_l(\cos\theta)\,d(\cos\theta)$. See G. N. Watson, "Theory of Bessel Functions," rev. ed., p. 128, St. Martin's Press, Inc., New York, 1944, and L. I. Schiff, "Quantum Mechanics," 2d ed., p. 77, McGraw-Hill Book Company. Inc., New York, 1955.

and of the field equations

$$\int_0^\infty dr\, r^2 U_k^l(r) U_{k'}^l(r) = \frac{1}{k^2 - k'^2}\int_0^\infty dr\left[v_k^l(r)\frac{d^2}{dr^2}v_{k'}^l(r) - v_{k'}^l(r)\frac{d^2}{dr^2}v_k^l(r)\right]$$

$$= \lim_{R\to\infty}\frac{1}{\pi}\frac{\sin R(k-k')}{k-k'} = \delta(k-k')$$

Furthermore, they and the spherical harmonics Y_l^m form a complete set of three-dimensional functions in the sense that

$$\int_0^\infty dk \sum_{l,m} U_k^l(r) U_k^l(r') Y_l^{m*}(\theta_r,\varphi_r) Y_l^m(\theta_{r'},\varphi_{r'}) = \delta^3(\mathbf{r}-\mathbf{r}')$$

Finally, if we introduce the expansion[1]

$$\phi(\mathbf{r},t) = \int_0^\infty \frac{dk}{(2\omega)^{\frac{1}{2}}} \sum_{l,m} U_k^l(r)[Y_l^m(\theta_r,\varphi_r)a_{lm}(k)e^{-i\omega t} + Y_l^{m*}(\theta_r,\varphi_r)a_{lm}^\dagger(k)e^{i\omega t}] \tag{5.10a}$$

then, by comparing with the continuum limit of (4.8), we see that $a_{lm}(k)$ is defined by

$$a_{lm}(k) = i^l\int d\Omega_k\, k Y_l^{m*}(\theta_k,\varphi_k)a(\mathbf{k})$$

With the help of

$$\delta^3(\mathbf{k}-\mathbf{k}') = \frac{\delta(k-k')}{k^2}\frac{\delta(\theta-\theta')\delta(\varphi-\varphi')}{\sin\theta}$$

we thus deduce the commutation relations

$$\begin{aligned} [a_{lm}(k), a_{l'm'}^\dagger(k')] &= \delta_{ll'}\delta_{mm'}\delta(k-k') \\ [a_{lm}(k), a_{l'm'}(k')] &= [a_{lm}^\dagger(k), a_{l'm'}^\dagger(k')] = 0 \end{aligned} \tag{5.11}$$

We can readily verify that (5.11) results in the correct commutation properties of $\phi(\mathbf{r},t)$.

In analogy to our plane-wave expansion for $\phi(\mathbf{r},t)$, we may also derive the continuous k-variable development given above as the limit of a discrete set which is selected by the boundary condition (5.2). From the asymptotic expansion of v, which is valid only for n or l much less than kR (e.g., for fixed n and l in the limit $R \to \infty$), we see that

$$U_k^l(R) = 0$$

requires $k = (n + l/2)\pi/R$ with $n = 0, 1, 2, \ldots.$ Under these conditions we see that

$$\sum_k \to \frac{R}{\pi}\int_0^\infty dk$$

---

[1] We have here used a convention which we shall keep henceforth. It relates discrete k-space quantities to continuous k-space ones. In the former, destruction operators are written as $a_\mathbf{k}$ or a_{klm}, whereas in the latter they appear as $a(\mathbf{k})$ or $a_{lm}(k)$.

and we can rewrite the above development as

$$\phi(\mathbf{r},t) = \sum_{k,l,m} \left(\frac{\pi}{2\omega R}\right)^{\frac{1}{2}} U_k^l(r)[Y_l^m(\theta_r,\varphi_r)a_{klm}e^{-i\omega t} + Y_l^{m*}(\theta_r,\varphi_r)a_{klm}^\dagger e^{i\omega t}]$$
$$[a_{klm},a_{k'l'm'}^\dagger] = \delta_{ll'}\delta_{mm'}\delta_{kk'} \qquad (5.10b)$$

with $a_{klm} \rightarrow (\pi/R)^{\frac{1}{2}}a_{lm}(k)$. In future developments we shall use both the discrete and the continuum form. To obtain numerical results from the theory, the latter is obviously more convenient. For easy reference, the relevant formulas relating discrete to continuum k space are collected in the Appendix.

In terms of the operators introduced in (5.10b), we get for the observables of interest[1]

$$H - E_0 = \sum_{l,m,k} a_{k,l,m}^\dagger a_{k,l,m}\omega \qquad (5.12)$$

and for L_3, by making use of $\partial Y_l^m(\theta_r,\varphi_r)/\partial\varphi_r = im\,Y_l^m(\theta_r,\varphi_r)$, we get

$$L_3 = \sum_{l,m,k} a_{k,l,m}^\dagger a_{k,l,m}m \qquad (5.13)$$

The vacuum is defined in terms of the new variables by

$$a_{k,l,m}\,|\,0\rangle = 0 \qquad (5.14)$$

In the limit of $R \rightarrow \infty$, in which case the boundary conditions (5.2) should be equivalent to the ones used in the momentum representation, (5.14) is identical with our old definition, since the new operators a are then linear combinations of the old ones. In the new representation, eigenstates of particles with energies ω are obtained by applying $a_{k,l,m}^\dagger$ to the vacuum. Thus

$$|\,n_{k_1l_1m_1},n_{k_2l_2m_2}\cdots\rangle = (a_{k_1l_1m_1}^\dagger)^{n_{k_1l_1m_1}}(a_{k_2l_2m_2}^\dagger)^{n_{k_2l_2m_2}}\cdots\,|\,0\rangle$$

We recognize from (5.13) that these states will also be eigenstates of L_3 belonging to the eigenvalue $\sum_{l_im_ik_i} m_in_{k_il_im_i}$. The $n_{k_il_im_i}$, e.g., the eigenvalues of $a_{klm}^\dagger a_{klm}$, are the numbers of particles with energies ω_i and angular momentum l_i with three-component m_i, so that L_3 has the integers as eigenvalues. To build up eigenstates of a given total angular momentum, we must properly combine the single-particle eigenstates we have constructed with the methods familiar from elementary wave mechanics.[2] We shall not go into this at present, but we shall carry out equivalent manipulations later. That there are several ways of constructing eigenstates of H is connected with the fact that in the limit $R \to \infty$ or $L \to \infty$, H is infinitely degenerate. The eigenstates of L_3 and H are just superpositions of eigenstates of $\mathbf{P}$ with the same eigenvalue of H, the coefficients being those which transform plane waves into spherical waves.

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[1] In L_3, a term $\sum_m \frac{1}{2}m$, which appears for the Klein-Gordon field, is zero because of symmetry between $+m$ and $-m$.

[2] L. D. Landau and E. M. Lifshitz, "Quantum Mechanics," chap. IV, Addison-Wesley Publishing Company, Reading, Mass., 1958.

5.2. Parity. A further constant of motion emerges from the invariance of H under the noncontinuous orthogonal transformation of the coordinates represented by the reflection $\mathbf{r} \to -\mathbf{r}$. Because this transformation cannot be generated by continuous rotations,[1] the constant associated with it, called parity, is independent of angular momentum. It is deduced by the usual argument. Since the substitution $\phi(\mathbf{r},t) \to \phi(-\mathbf{r},t)$ leaves the commutation relations invariant, there must be a unitary transformation effecting the substitution:

$$\mathscr{P}_+\phi(\mathbf{r},t)\mathscr{P}_+^{-1} = \phi(-\mathbf{r},t)$$

$$\mathscr{P}_+\mathscr{P}_+^\dagger = \mathscr{P}_+^\dagger\mathscr{P}_+ = 1 \tag{5.14a}$$

Also, H is invariant under the substitution $\phi(\mathbf{r},t) \to \phi(-\mathbf{r},t)$, so that we have

$$\mathscr{P}_+H\mathscr{P}_+^{-1} = H \qquad [\mathscr{P}_+,H] = 0$$

which implies that $\mathscr{P}_+$ is a constant. However, both H and the commutation relations are also invariant under $\phi(\mathbf{r},t) \to -\phi(-\mathbf{r},t)$, so that one can also define a reflection

$$\mathscr{P}_-\phi(\mathbf{r},t)\mathscr{P}_-^{-1} = -\phi(-\mathbf{r},t) \tag{5.14b}$$

and $\mathscr{P}_-$ is also constant. Only when there is an interaction can we tell which is the right reflection property of ϕ, that is to say, which of the two operators $\mathscr{P}_\pm$ is a constant. For instance, if H includes a term

$$\int d^3r\ \rho(\mathbf{r})\phi(\mathbf{r},t)$$

where $\rho(\mathbf{r})$ is invariant under reflections, then only $\mathscr{P}_+$ commutes with H and ϕ is then called a scalar. On the other hand, a term

$$\int d^3r\ \rho(\mathbf{r})\boldsymbol{\sigma}\cdot\nabla\phi(\mathbf{r},t) \qquad \text{with} \qquad \mathscr{P}_\pm\boldsymbol{\sigma}\mathscr{P}_\pm^{-1} = \boldsymbol{\sigma}$$

commutes only with $\mathscr{P}_-$, and ϕ is then called a pseudoscalar. This latter case is realized in nature by the pion field.

The operators $\mathscr{P}_\pm$ can be diagonalized in an angular-momentum rather than in a momentum representation since $[\mathscr{P}_\pm,\mathbf{L}] = 0$ but $\mathscr{P}_\pm\mathbf{P}\mathscr{P}_\pm^{-1} = -\mathbf{P}$. Since

$$Y_l^m(-\mathbf{r}) \equiv Y_l^m(\pi - \theta_r, \varphi_r + \pi) = (-1)^l Y_l^m(\theta_r,\varphi_r) = (-1)^l Y_l^m(\mathbf{r})$$

---

[1] This means that $\mathscr{P}$ has no classical analogue; there is no infinitesimal generator for a reflection.

we find as explicit expressions[1]

$$\begin{aligned}\mathcal{P}_+ &= \exp\,(-i\pi \sum_{klm} l a^\dagger_{klm} a_{klm}) \\ \mathcal{P}_- &= \exp\,[-i\pi \sum_{klm} (l+1) a^\dagger_{klm} a_{klm}]\end{aligned} \tag{5.14c}$$

and from these expressions we deduce

$$\mathcal{P}_+ a_{klm} \mathcal{P}_+^{-1} = (-1)^l a_{klm} \qquad \mathcal{P}_- a_{klm} \mathcal{P}_-^{-1} = (-1)^{l+1} a_{klm}$$

From (5.14c) it appears that $\mathcal{P}_+$ is -1 raised to the number of particles with odd angular momentum, and $\mathcal{P}_-$ is -1 raised to the number of particles with even angular momentum. Hence parity is a multiplicative quantity; for several particles it is the product of the individual parities. Scalar particles have only their orbital parity $(-)^l$, whereas pseudoscalar particles also have an intrinsic negative parity.

5.3. Number of Particles and Particle Density. Another observable which commutes with H but has no analogy in classical mechanics is the number of particles[2]

$$\begin{aligned}N &= \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \\ &= \sum_{klm} a^\dagger_{klm} a_{klm}\end{aligned} \tag{5.15}$$

Its eigenvalues are the sums over the integers $n_{\mathbf{k}}$ (or n_{klm}) which we interpreted as the number of particles present in a state with momentum **k** (or angular-momentum z component m):

$$N \mid n_{\mathbf{k}_j} \rangle = \sum_{\mathbf{k}_j} n_{\mathbf{k}_j} \mid n_{\mathbf{k}_j} \rangle \tag{5.16}$$

Thus N can be called the operator for the total number of particles present. We obviously have

$$[H,N] = 0 \tag{5.17}$$

which means that no particles are created or destroyed. In fact, if we define the operator for the number of particles of a given momentum **k** as $N_{\mathbf{k}} = a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$, so that $N = \sum_{\mathbf{k}} N_{\mathbf{k}}$, then we find that

$$[N_{\mathbf{k}},H] = 0 \qquad \dot{N}_{\mathbf{k}} = 0$$

---

[1] Note that $e^{i\mathcal{O}} a e^{-i\mathcal{O}}$ is defined by expanding the exponentials

$$e^{i\mathcal{O}} a e^{-i\mathcal{O}} = a + i[\mathcal{O},a] + \frac{i^2}{2!}[\mathcal{O},[\mathcal{O},a]] + \cdots$$

so that, if $[a,a^\dagger] = 1$,

$$e^{-i\pi a^\dagger a} a e^{i\pi a^\dagger a} = -a$$

The phase factor in $\mathcal{P}_\pm$, which is left open by (5.14a) and (5.14b), is chosen by $\mathcal{P}_\pm \mid 0 \rangle = \mid 0 \rangle$. By means of (5.14$a$) we can also compute $\mathcal{P}_\pm a_{\mathbf{k}} \mathcal{P}_\pm^{-1} = \pm a_{-\mathbf{k}}$.

[2] This equation and related equations given later are always to be understood in the limit $L \to \infty$ or $R \to \infty$.

This tells us that no particles are transferred from one momentum state to another; in other words, no particles are scattered by the Hamiltonian we have been considering. Equation (5.17) will no longer hold for systems that we shall consider later on.

Like the operators considered previously, N can be expressed as a volume integral. If we decompose ϕ into a positive- and a negative-frequency part,

$$\begin{aligned}
\phi(\mathbf{r},t) &= \phi^{(+)}(\mathbf{r},t) + \phi^{(-)}(\mathbf{r},t) \\
\phi^{(+)}(\mathbf{r},t) &= \frac{1}{L^{\frac{3}{2}}} \sum_{\mathbf{k}} \frac{e^{(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{(2\omega)^{\frac{1}{2}}}\, a_{\mathbf{k}} \\
\phi^{(-)} &= \phi^{(+)\dagger} \qquad\qquad (5.18) \\
[\phi^{(+)}(\mathbf{r},t),\dot{\phi}^{(+)}(\mathbf{r}',t')]_{t=t'} &= [\phi^{(-)}(\mathbf{r},t),\, \dot{\phi}^{(-)}(\mathbf{r}',t')]_{t=t'} \\
&= [\phi^{(+)}(\mathbf{r},t),\phi^{(+)}(\mathbf{r}',t')]_{t=t'} \\
&= [\phi^{(-)}(\mathbf{r},t),\, \phi^{(-)}(\mathbf{r}',t')]_{t=t'} = 0 \\
[\phi^{(+)}(\mathbf{r},t),\dot{\phi}^{(-)}(\mathbf{r}',t')]_{t=t'} &= [\phi^{(-)}(\mathbf{r},t),\, \dot{\phi}^{(+)}(\mathbf{r}',t')]_{t=t'} = \frac{i\delta^3(\mathbf{r}-\mathbf{r}')}{2}
\end{aligned}$$

we find

$$N = -i\int d^3r\, [\dot{\phi}^{(-)}\phi^{(+)} - \phi^{(-)}\dot{\phi}^{(+)}] \tag{5.19}$$

In the nonrelativistic limit this becomes for the Schrödinger field the more familiar expression

$$N = \int d^3x\, \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t) \tag{5.20}$$

In elementary wave mechanics this is put equal to 1, which means, in our present language, that there we consider only one-particle states.

We can also show that Ehrenfest's theorem[1] of wave mechanics holds in our general theory. If, in analogy with wave mechanics, we define the center of mass by the operator[2]

$$\mathbf{R} = \frac{1}{N}\int d^3r\, \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t)\mathbf{r} \tag{5.21}$$

we obtain, with the help of (4.22), (5.3), and partial integrations,

$$\dot{\mathbf{R}} = i[H,\mathbf{R}] = \frac{\mathbf{P}}{Nm} \tag{5.22}$$

[1] See L. I. Schiff, "Quantum Mechanics," 2d ed., p. 25, McGraw-Hill Book Company, Inc., New York, 1955.

[2] Note that $[R_i,P_j] = i\delta_{ij}$, as in elementary wave mechanics.

Thus the total momentum is equal to the total mass multiplied by the velocity of the center of mass. The relativistic analogy of (5.22) holds only for the center of energy,

$$\mathbf{R} = \frac{1}{E}\int d^3r\, \tfrac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2]\mathbf{r} \tag{5.23}$$

$$\dot{\mathbf{R}} = \frac{\mathbf{P}}{E} \tag{5.24}$$

5.4. Local Observables. The observables considered so far were of the form of an integral over all space. This suggests interpreting the integrand as the corresponding local density and an integral over a finite volume as that part of the observable contained in this volume. However, the quantities integrated over the whole volume L^3 may fail to commute with ϕ or bilinear operators such as the momentum density $\mathbf{P}(\mathbf{r})$, and hence the states considered so far will in general not be eigenstates of local quantities such as the momentum density. This will become quite clear in the next chapter, in which we consider states.

With respect to local quantities, there is an important difference, which we shall now consider, between the relativistic and nonrelativistic case. If we define the number of particles in a volume v as

$$N_v(t) = \int_v d^3r\, N(\mathbf{r},t) \tag{5.25}$$

then, for the nonrelativistic field considered,

$$N(\mathbf{r},t) = \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t) \tag{5.26}$$

and for the relativistic one,

$$N(\mathbf{r},t) = -i[\dot{\phi}^{(-)}(\mathbf{r},t)\phi^{(+)}(\mathbf{r},t) - \phi^{(-)}(\mathbf{r},t)\dot{\phi}^{(+)}(\mathbf{r},t)] \tag{5.27}$$

In the nonrelativistic case, it follows from the commutation relations (4.22) that

$$[N_{v_1}(t), N_{v_2}(t')]_{t=t'} = 0 \tag{5.28}$$

whether the volumes v_1 and v_2 overlap or not. This means that we can talk of a definite number (e.g., 1) of nonrelativistic particles in a volume of any size, no matter how small or how large. It is true that this number does not remain constant, since

$$[N_v(t),H] = -\frac{1}{2m}\int_v d^3r\, [\psi^\dagger(\mathbf{r},t)\nabla^2\psi(\mathbf{r},t) - \nabla^2\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t)] \neq 0$$

and the surface integral to which this reduces is finite for finite volumes v, but this only means that the wave packet for a localized particle spreads out as time goes on. For the relativistic field ϕ, the commutation relation (5.28) does not hold even if the volumes v_1 and v_2 do not overlap. This comes about because of the factor ω^{-1}, which spoils the vanishing of the commutator

$$[\phi^{(+)}(\mathbf{r},t),\ \phi^{(-)}(\mathbf{r}',t')]_{t=t'} = \mathop{\mathrm{S}}_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{2\omega} = \tfrac{1}{2}\Delta^{(+)}(\mathbf{r}-\mathbf{r}') \qquad (5.29a)$$

when $\mathbf{r} \neq \mathbf{r}'$.

The behavior of the commutator can be found as follows:

$$\Delta^{(+)}(\mathbf{r}) = \frac{1}{8\pi^3}\int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(k^2+m^2)^{\frac{1}{2}}} = \frac{1}{2\pi^2 r}\int k\,dk \frac{\sin kr}{(k^2+m^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} dk \frac{e^{ikr}}{(k^2+m^2)^{\frac{1}{2}}} \qquad (5.29b)$$

Substitution of $k = m\sinh\theta$ gives

$$\Delta^{(+)}(\mathbf{r}) = -\frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\int e^{imr\sinh\theta}\,d\theta = -\frac{i}{4\pi r}\frac{\partial}{\partial r}H_0^{(1)}(imr)$$

$$= -\frac{m}{4\pi r}H_1^{(1)}(imr)$$

We see, therefore, that the commutator behaves like a Hankel function of the first kind, $H^{(1)}(imr)$, which has the following properties:[1]

$$\lim_{r\to 0}\Delta^{(+)}(\mathbf{r}) = \lim_{r\to 0}\frac{1}{2\pi r^2}$$

$$\lim_{r\to\infty}\Delta^{(+)}(\mathbf{r}) = \frac{m}{4\pi r}\left(\frac{2}{\pi m r}\right)^{\frac{1}{2}} e^{-mr}$$

Thus, the dominant behavior for asymptotically large distances arises in (5.29b) from the exponential e^{ikr}, evaluated at the complex pole $k = im$.

Correspondingly, we find for the commutator of the local density $N(\mathbf{r},t)$ with that at another spatial point, $\mathbf{r}'$, but at the same time t,

$$[N(\mathbf{r},t),\ N(\mathbf{r}',t')]_{\substack{t=t'\\ \mathbf{r}\neq\mathbf{r}'}} = [\dot\phi^{(-)}(\mathbf{r}',t)\dot\phi^{(+)}(\mathbf{r},t) - \phi^{(-)}(\mathbf{r},t)\dot\phi^{(+)}(\mathbf{r}',t)]\frac{\Delta^{(+)}(\mathbf{r}-\mathbf{r}')}{2}$$

$$+ [\phi^{(-)}(\mathbf{r}',t)\phi^{(+)}(\mathbf{r},t) - \phi^{(-)}(\mathbf{r},t)\phi^{(+)}(\mathbf{r}',t)]\frac{\Gamma^{(+)}(\mathbf{r}-\mathbf{r}')}{2}$$

with $$\Gamma^{(+)}(\mathbf{r}-\mathbf{r}') = 2[\dot\phi^{(+)}(\mathbf{r},t),\ \dot\phi^{(-)}(\mathbf{r}',t)] = \mathop{\mathrm{S}}_{\mathbf{k}}\omega e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$= \frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} dk\, e^{ikr}(k^2+m^2)^{\frac{1}{2}} \qquad r = |\mathbf{r}-\mathbf{r}'|$$

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[1] G. N. Watson, "A Treatise on the Theory of Bessel Functions," 2d ed., chaps. 3, 6, 7, Cambridge University Press, New York, 1958. Note that

$$H_0^{(1)}(imr) = \frac{2}{\pi i}K_0(mr)$$

Here we used the algebraic identity

$$[AB,CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$$

and

$$[\phi^{(+)}(\mathbf{r},t), \dot{\phi}(\mathbf{r}',t)] = \tfrac{1}{2}i\delta^3(\mathbf{r} - \mathbf{r}')$$

The integral for $\Gamma^{(+)}$ diverges in its present form, but it can be made convergent by bringing down a sufficient number of powers of k in the denominator by differentiation with respect to r. For large distances $|\mathbf{r} - \mathbf{r}'|$ the dominant behavior of $\Gamma^{(+)}$ is again determined by the exponential evaluated at $k = im$. Thus, for the relativistic field considered, the commutator of the local density $N(\mathbf{r},t)$ with that at another spatial point but at the same time [e.g., $N(\mathbf{r}',t)$] goes to zero only if the two points are separated by a distance $|\mathbf{r} - \mathbf{r}'| \gg m^{-1}$. The same statement holds for the number of particles contained in two nonoverlapping volumes v_1 and v_2, as shown in Fig. 5.1. Here m^{-1} is the Compton wavelength of the field particle, and for the π meson, for instance, it is $\sim 10^{-13}$ cm. Hence it is not possible to assert that one pion (or any other definite number) is in a volume the boundary of which is defined within the order of 10^{-13} cm or less. That would be true only if this state were an eigenstate of N_v with eigenvalue 1 for this particular volume and with eigenvalue 0 for all neighboring volumes within 10^{-13} cm. Because of the noncommutativity of such closely neighboring N_v, this is impossible. The best we can do relativistically is to have eigenvalue 0 for those N_v for which v is many m^{-1} apart from the volume which contains the particle. Physically, this is connected with the fact that defining the boundary so sharply, $\Delta r \ll m^{-1}$, requires that there be an external field partially composed of wavelengths $< m^{-1}$. Such a field is capable of creating new particles. Because of the identity of particles in field theory, the new particles cannot be distinguished from the old ones. Hence the state will cease to be a one-particle state.

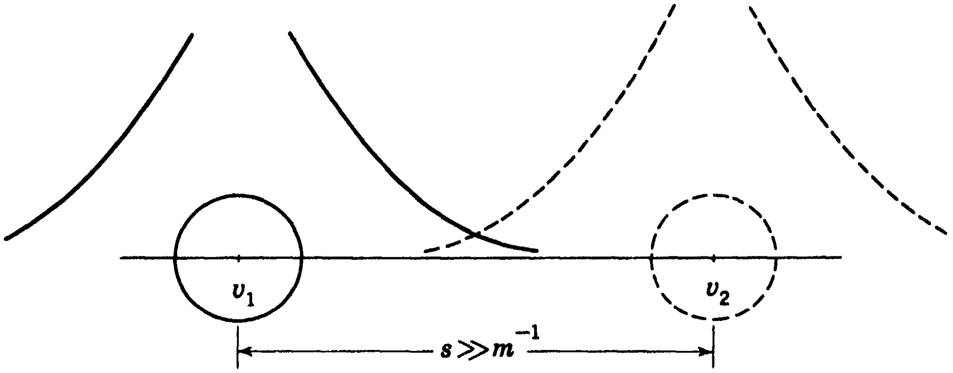

Fig. 5.1. Distribution of mesons about two nonoverlapping volumes v_1 and v_2 separated by a distance s much larger than the Compton wavelength m^{-1} of the particles.

It appears that the fundamental principles of relativity ($E = mc^2$) and quantum theory ($E = h\nu$) give an important modification to our concepts of particles. Whereas in the nonrelativistic limit they appear as points and there is no lower limit to the size of the region into which they can be confined, in relativistic field theory the quanta of the field have roughly the size of their Compton wavelength. This is the origin of the decrease of the electromagnetic interactions when wavelengths $\lambda < m^{-1}$ are involved. An electron, for instance, acts like a charged sphere with radius $r \sim m_e^{-1}$, and the effect of smaller wavelengths is averaged out. Hence the cross sections for scattering of photons by electrons decrease for photon wavelengths $<m_e^{-1}$.¶ Similarly, this effect decreases the binding of the hydrogenic S electron, since its size does not permit it to take full advantage of the narrow singular part of the Coulomb potential.

Summarizing, we can say that the behavior of observables in quantum field theory is like that of an ensemble of free particles. The question of the size of the particles and other features of local quantities will be further illuminated when we discuss typical states in the next chapter.

¶ See W. Thirring, "Principles of Quantum Electrodynamics," Academic Press, Inc., New York, 1958.

CHAPTER 6

States

6.1. Vacuum and One-particle States. The states we have been mainly interested in so far have been eigenstates of the energy. The state with the lowest energy, $|\,0\rangle$, has no particles and, appropriately, is called the vacuum. Application of any of the $a_{\mathbf{k}}^{\dagger}$ to $|\,0\rangle$ creates a state with one particle present with momentum **k**. The most general one-particle state is obtained by multiplying $|\,0\rangle$ with a general linear combination of operators $a_{\mathbf{k}}^{\dagger}$ with different values of **k**. This can also be done by means of the field variables $\phi^{(-)}(\mathbf{r},t)$ of Eq. (5.18) or, in the nonrelativistic case, by $\psi^{\dagger}(\mathbf{r},t)$.

Treating the latter and more familiar case first, we can write

$$|\,1\rangle = \int d^3r\, f(\mathbf{r},t)\psi^{\dagger}(\mathbf{r},t)\,|\,0\rangle \tag{6.1}$$

We note that our previous one-particle states $a_{\mathbf{k}}^{\dagger}\;|\,0\rangle$ or $a_{klm}^{\dagger}\;|\,0\rangle$ are special cases of (6.1), with

$$f(\mathbf{r},t) = \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-E_k t)}}{L^{\frac{3}{2}}}$$

or

$$f(\mathbf{r},t) \propto U_k^l(r)Y_l^m(\theta_r,\varphi_r)e^{-iE_k t}$$

since these states are time-independent if they are eigenstates of the Hamiltonian. The normalization of the one-particle state (6.1), $\langle 1\,|\,1\rangle = 1$, requires

$$\begin{aligned}\langle 1\,|\,1\rangle &= \langle 0\,|\int f^*(\mathbf{r},t)\psi(\mathbf{r},t)\,d^3r\int d^3r'\,f(\mathbf{r}',t)\psi^{\dagger}(\mathbf{r}',t)\,|\,0\rangle\\ &= \int d^3r\,f^*(\mathbf{r},t)f(\mathbf{r},t)\\ &= \int d^3r\,f^*(\mathbf{r})f(\mathbf{r}) = 1\end{aligned} \tag{6.2}$$

This is the normalization condition for the wave function in wave mechanics, and, indeed, f plays the role of this quantity. It appears whenever expectation values of a quantity like the energy density $H(\mathbf{r})$, the momentum density $\mathbf{P}(\mathbf{r})$, or the density of the number of particles $N(\mathbf{r})$ are computed.[1] Thus

$$\begin{aligned}\langle 1 \mid N(\mathbf{r}) \mid 1\rangle &= \langle 0 \mid \int f^*(\mathbf{r}')\psi(\mathbf{r}')\, d^3r'\, \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \int f(\mathbf{r}'')\psi^\dagger(\mathbf{r}'')\, d^3r'' \mid 0\rangle \\ &= f^*(\mathbf{r})f(\mathbf{r}) \\ \langle 1 \mid H(\mathbf{r}) \mid 1\rangle &= \frac{1}{2m} \langle 0 \mid \int f^*(\mathbf{r}')\psi(\mathbf{r}')\, d^3r'\, \nabla\psi^\dagger(\mathbf{r}) \cdot \nabla\psi(\mathbf{r}) \int f(\mathbf{r}'')\psi^\dagger(\mathbf{r}'')\, d^3r'' \mid 0\rangle \\ &= \frac{1}{2m} \nabla f^*(\mathbf{r}) \cdot \nabla f(\mathbf{r})\end{aligned}$$

and, similarly,

$$\langle 1 \mid \mathbf{P}(\mathbf{r}) \mid 1\rangle = \frac{1}{2i} [f^*(\mathbf{r})\nabla f(\mathbf{r}) - f(\mathbf{r})\nabla f^*(\mathbf{r})] \tag{6.3}$$

We shall now investigate whether the field quanta can be considered particles in the sense that they are objects localized in a certain region in space. As in wave mechanics, we can at a certain time have a particle density with an arbitrary spatial distribution. To be sure, such a state is not, in general, an eigenstate of energy and momentum, but this may also be true in wave mechanics where a localized wave packet eventually diffuses. Our nonrelativistic particles need not have a finite size at a given time t, since we can have a state for which $f(\mathbf{r},t)$ is different from zero only in an arbitrarily small region [e.g., $f(\mathbf{r},0) = \delta^3(\mathbf{r})$]. In this case the expectation values of all densities will, according to (6.3), be zero outside this region. We see from (4.22) that such a state is even an eigenstate of densities outside this region belonging to the eigenvalue 0. This means that there are states for which, outside a region as tiny as we like, no experiment will find any trace of a particle. Nevertheless, we shall always have

$$N \mid 1\rangle = \mid 1\rangle \tag{6.4}$$

In particular, at $t = 0$, say, $\mid 1\rangle = \psi^\dagger(\mathbf{r}) \mid 0\rangle$ is easily seen to be an eigenstate of N_v (although not normalized) belonging to the eigenvalue 1 if v contains $\mathbf{r}$ and to eigenvalue 0 if it does not.[2] To show this, we use

$$[N_v, \psi^\dagger(\mathbf{r})] = \psi^\dagger(\mathbf{r}) \int d^3r'\, \delta^3(\mathbf{r} - \mathbf{r}')$$

which also proves that N_v has integral eigenvalues for arbitrary volumes v.

---

[1] $H(\mathbf{r})$, $\mathbf{P}(\mathbf{r})$, and $N(\mathbf{r})$ are the integrands of the corresponding integrated observables, evaluated at $t = 0$.

[2] This state is not an eigenstate of H, since $[\psi^\dagger(\mathbf{r},t), H] \neq 0$ and it will thus be time-dependent. When no time dependence is indicated for N_v, we mean $N_v(0)$.

The relativistic field states behave differently. First of all, in this case the vacuum is not even an eigenstate of the local densities $H(\mathbf{r})$, $\mathbf{P}(\mathbf{r})$, or $\mathbf{L}(\mathbf{r})$. These quantities contain terms proportional to $\phi^{(-)2}$ and therefore lead from the vacuum to a two-particle state. Nor is the vacuum expectation value of $H(\mathbf{r})$ equal to zero:

$$\begin{aligned}\langle 0 \mid H(\mathbf{r}) \mid 0 \rangle &= \tfrac{1}{2}\langle 0 \mid \dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 \mid 0 \rangle \\ &= \tfrac{1}{2}\langle 0 \mid \dot{\phi}^{(+)}(\mathbf{r})\dot{\phi}^{(-)}(\mathbf{r}) + \nabla\phi^{(+)}(\mathbf{r}) \cdot \nabla\phi^{(-)}(\mathbf{r}) \\ &\qquad + m^2\phi^{(+)}(\mathbf{r})\phi^{(-)}(\mathbf{r}) \mid 0 \rangle \\ &= \frac{1}{L^3}\sum_{\mathbf{k}} \tfrac{1}{2}\omega\end{aligned}$$

We may, however, redefine the densities so as to ensure the vanishing of their vacuum expectation values. This is accomplished by writing all $\phi^{(+)}$ operators to the right of the $\phi^{(-)}$ ones, e.g.,

$$\begin{aligned}\phi^{(+)}\phi^{(-)} &\rightarrow \phi^{(-)}\phi^{(+)} \\ \phi^{(+)}\dot{\phi}^{(-)} &\rightarrow \dot{\phi}^{(-)}\phi^{(+)}\end{aligned} \tag{6.5}$$

so that the momentum density, for example, becomes

$$\begin{aligned}\mathbf{P}(\mathbf{r}) = -[\dot{\phi}^{(-)}(\mathbf{r})\nabla\phi^{(-)}(\mathbf{r}) &+ \dot{\phi}^{(-)}(\mathbf{r})\nabla\phi^{(+)}(\mathbf{r}) \\ &+ \nabla\phi^{(-)}(\mathbf{r})\dot{\phi}^{(+)}(\mathbf{r}) + \dot{\phi}^{(+)}(\mathbf{r})\nabla\phi^{(+)}(\mathbf{r})]\end{aligned}$$

It should be noted that this does not take care of the $\phi^{(-)2}$ terms, so that the vacuum is still not an eigenstate of local densities. However, the above rearrangement does eliminate the zero-point energy for E. These alterations only change the observables by ordinary (although infinite) nonmeasurable numbers. Furthermore, these numbers are real, so that the hermiticity of the observables is not destroyed. Henceforth we shall always assume ordered products for observables quadratic in ϕ. This does not mean that the vacuum fluctuations of ϕ vanish. Thus $(\Delta\phi)^2 = \langle 0 \mid \phi^2 \mid 0 \rangle$ is still given by (4.13), and the fluctuation in the energy density $H(\mathbf{r})$, with the reordering for H (but not for H^2), is also different from zero, $[\Delta H(\mathbf{r})]^2 = \langle 0 \mid H^2(\mathbf{r}) \mid 0 \rangle$, and diverges even faster than that of the field operator ϕ. That is to say, in a relativistic theory we are never sure that the local energy is zero. This arises again from the fact that the accurate definition of a volume requires high momenta and energies which, in a relativistic theory, may create particles. As we go along, we shall notice that the virtual existence of particles throughout space is a most striking feature of this theory.

The one-particle states of the relativistic theory also present interesting features. To write our arbitrary linear combination of the creation operators $a_{\mathbf{k}}^{\dagger}$ in momentum space, $\sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} f_{\mathbf{k}}$, in terms of the variables $\phi^{(-)}$, we have to introduce $F_{\mathbf{k}} = (2\omega)^{\frac{1}{2}} f_{\mathbf{k}}$ and its Fourier transform $F(\mathbf{r})$, because

$$\frac{a_{\mathbf{k}}^{\dagger}}{(2\omega)^{\frac{1}{2}}} = \frac{1}{L^{\frac{3}{2}}} \int \phi^{(-)}(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r$$

Thus, at $t = 0$, we have

$$|\,1\rangle = \sum_{\mathbf{k}} f_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} \,|\, 0\rangle = \int d^3r\, F(\mathbf{r}) \phi^{(-)}(\mathbf{r}) \,|\, 0\rangle \tag{6.6}$$

The normalization condition

$$\sum_{\mathbf{k}} |\, f_{\mathbf{k}}\,|^2 = \sum_{\mathbf{k}} \frac{|F_{\mathbf{k}}|^2}{2\omega} = 1 \tag{6.7}$$

does not correspond to the usual $\int d^3r\, |F(\mathbf{r})|^2 = 1$ but rather to

$$\langle 0 \,|\int d^3r\, d^3r'\, F^*(\mathbf{r}) F(\mathbf{r}') \phi^{(+)}(\mathbf{r}) \phi^{(-)}(\mathbf{r}') \,|\, 0\rangle$$
$$= \tfrac{1}{2} \int F^*(\mathbf{r}) \Delta^{(+)}(\mathbf{r} - \mathbf{r}') F(\mathbf{r}')\, d^3r\, d^3r' = 1$$

This is due to the factor ω, which, even with the redefinition (6.5), makes it impossible to find a spatial distribution $F(\mathbf{r})$ such that the expectation values of all densities are zero outside a certain region. For instance, putting all $f_{\mathbf{k}}$ equal to $1/L^{\frac{3}{2}}$, corresponding to a spatial δ function at $\mathbf{r} = 0$ for $f(\mathbf{r})$, but not for $F(\mathbf{r})$, we obtain for the particle density, Eq. (5.27),

$$\langle 1 \,|\, N(\mathbf{r}) \,|\, 1\rangle = \sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}'} \frac{1}{2} \left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}} + \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}} \right] \frac{e^{i\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}')}}{L^6} \langle 0 \,|\, a_{\mathbf{q}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}'} a_{\mathbf{q}'}^{\dagger} \,|\, 0\rangle$$
$$= \mathop{S}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \omega^{\frac{1}{2}} \mathop{S}_{\mathbf{k}'} e^{-i\mathbf{k}'\cdot\mathbf{r}} \omega'^{-\frac{1}{2}} \tag{6.8}$$

This is clearly not equal to zero for $\mathbf{r} \neq 0$, so that the one-particle state so described cannot be considered localized; it is, rather, spread out over a distance of the order of $1/m$. On the other hand, if we choose $F_{\mathbf{k}}$ equal to $1/L^{\frac{3}{2}}$, corresponding to $F(\mathbf{r}) = \delta^3(\mathbf{r})$, then we find for $N(\mathbf{r})$, by means of the commutation relations (5.18) and (5.29*a*),

$$\langle 1 \,|\, N(\mathbf{r}) \,|\, 1\rangle = -i \langle 0 \,|\, \phi^{(+)}(0) [\dot{\phi}^{(-)}(\mathbf{r}) \phi^{(+)}(\mathbf{r}) - \phi^{(-)}(\mathbf{r}) \dot{\phi}^{(+)}(\mathbf{r})] \phi^{(-)}(0) \,|\, 0\rangle$$
$$= \tfrac{1}{4} [\delta^3(\mathbf{r}) \Delta^{(+)}(\mathbf{r}) + \Delta^{(+)}(-\mathbf{r}) \delta^3(\mathbf{r})]$$
$$= \tfrac{1}{2} \delta^3(\mathbf{r}) \Delta^{(+)}(\mathbf{r})$$

This is indeed zero for $r \neq 0$, so that the state appears to be localized at the origin.[1] However, in this case the expectation value of energy density becomes

$$\begin{aligned}\langle 1 \mid H(\mathbf{r}) \mid 1 \rangle &= \tfrac{1}{2}\langle 0 \mid \phi^{(+)}(0)\{\dot{\phi}^2(\mathbf{r}) + [\nabla\phi(\mathbf{r})]^2 + m^2\phi^2(\mathbf{r})\}\phi^{(-)}(0) \mid 0 \rangle \\ &= \tfrac{1}{4}\{[\delta^3(\mathbf{r})]^2 + [\nabla\Delta^{(+)}(\mathbf{r})]^2 + [m\Delta^{(+)}(\mathbf{r})]^2\} \qquad (6.9)\end{aligned}$$

The dominant spatial dependence of this function is given by e^{-2mr} for $r \gg m^{-1}$. This behavior can be understood by considering a single particle located at the origin. Any measurement of the energy creates particles, but these have already been accounted for by the redefinition (6.5). This ensures that, in the absence of real (as opposed to virtual) particles, e.g., for the vacuum state, the expectation value of the energy density is zero. However, because of the presence of the particle at the origin, something new can happen if we measure nearby. A pair of particles may be created at a distance $r < m^{-1}$, one of which stays there, whereas the other annihilates the particle at the origin. The distance over which virtual particles can spread is limited by $\Delta E\, \Delta\tau \sim 1$. Since $\Delta E > 2m$ for creation of a pair of particles, $\Delta\tau < (2m)^{-1}$, and the particle cannot propagate further than $(2m)^{-1}$. Therefore the above type of event will influence the energy density within distances of the order of m^{-1} about the origin.

By similar calculations we recognize that N_v has only integer eigenvalues if $v^{1/3}$ has an extent much larger than the Compton wavelength of the field quanta. We find that $[N_v,\phi^{(-)}(\mathbf{r})] \neq \phi^{(-)}(\mathbf{r})$ but contains an additional term proportional to $\phi^{(-)}$ averaged around $\mathbf{r}$ within a Compton wavelength:

$$[N_v,\phi^{(-)}(\mathbf{r})] = \begin{cases} \tfrac{1}{2}[\phi^{(-)}(\mathbf{r}) - i\int_v \dot{\phi}^{(-)}(\mathbf{r}')\Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r'] & \text{if } \mathbf{r} \text{ is in } v \\ -\dfrac{i}{2}\displaystyle\int_v \dot{\phi}^{(-)}(\mathbf{r}')\Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' & \text{if } \mathbf{r} \text{ is not in } v \end{cases}$$

Hence $\phi^{(-)}(\mathbf{r}) \mid 0 \rangle$ will not be an eigenstate of N_v. Nevertheless, if v is much larger than m^{-3} and $F(\mathbf{r})$ is a smooth distribution in a volume $\gg m^{-3}$ about the middle of v, then $\int d^3r\, F(\mathbf{r})\phi^{(-)}(\mathbf{r}) \mid 0 \rangle$ will almost be an eigenstate of this N_v. Since the local energy densities at different points but at the same time commute,[2] $[H(\mathbf{r}), H(\mathbf{r}')] = 0$ if $\mathbf{r} \neq \mathbf{r}'$, it

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[1] Our choice of $F(\mathbf{r})$ does not lead to a normalized state $\mid 1 \rangle$. Note, furthermore, that the state described by $F(\mathbf{r}) = \delta^3(\mathbf{r})$ is not an eigenstate of $N(\mathbf{r})$, e.g.,

$$N(\mathbf{r}) \mid 1 \rangle \neq \text{constant} \mid 1 \rangle \qquad \text{for } \mathbf{r} = 0$$

consistent with our discussion in Chap. 5. However, the expectation value of $N(\mathbf{r})$ for this state is localized.

[2] Note that the ordering of $H(\mathbf{r})$ changes it only by an ordinary number, so that its commutation properties are not changed.

might seem possible to build states for which the energy is exactly localized. This can indeed be done, but then it can be shown that these states do not possess a definite number of particles. Local quantities and the number of particles are, therefore, complementary concepts.

6.2. Two-particle States. In order to get a feeling for some of the consequences of the Bose-Einstein statistics of the field quanta, we shall finally study interference and fluctuation phenomena. For the sake of simplicity, we shall keep to the nonrelativistic field. The results for the relativistic field ϕ are similar, but are complicated by the additional effects discussed above.

The interference effects already appear for the two-particle states. Nonrelativistically, the most general state of this type is[1]

$$| 2\rangle = \int d^3r_1\, d^3r_2 f(\mathbf{r}_1,\mathbf{r}_2)\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)\,|\,0\rangle \tag{6.10}$$

and belongs to the eigenvalue 2 of N. The normalization condition for (6.10) is

$$\langle 2\,|\,2\rangle = \int d^3r_1\, d^3r_2\, f^*(\mathbf{r}_1,\mathbf{r}_2)[f(\mathbf{r}_1,\mathbf{r}_2) + f(\mathbf{r}_2,\mathbf{r}_1)] = 1 \tag{6.11}$$

The second term in (6.11) arises because of the Bose statistics of the field quanta. In fact, we could have restricted ourselves to a symmetric f in (6.10), because the part of f which is odd in $\mathbf{r}_1$ and $\mathbf{r}_2$ does not contribute. Since $\psi^\dagger(\mathbf{r}_1)$ and $\psi^\dagger(\mathbf{r}_2)$ commute, $\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)$ is even in $\mathbf{r}_1$ and $\mathbf{r}_2$ and gives zero on integration with an odd function.

There are some peculiar features connected with these facts which are best illustrated by calculating the expectation value of $N(\mathbf{r})$. We shall do this for a two-particle state for which $f(\mathbf{r}_1,\mathbf{r}_2)$ is of the form $f_1(\mathbf{r}_1)f_2(\mathbf{r}_2)$, so that the particles would be independent could they be distinguished.[2] Furthermore, we assume that each $f_j(\mathbf{r}_j)$ is normalized to unity, e.g., $\int d^3r_j\, f_j(\mathbf{r}_j)^2 = 1$, so that the correctly normalized over-all distribution function f is

$$f(\mathbf{r}_1,\mathbf{r}_2) = \frac{f_1(\mathbf{r}_1)\, f_2(\mathbf{r}_2)}{[1 + |(f_1,f_2)|^2]^{\frac{1}{2}}} \tag{6.12}$$

with

$$(f_i,f_j) \equiv \int d^3r\, f_i^*(\mathbf{r})f_j(\mathbf{r}) \tag{6.13}$$

---

[1] If this state is an eigenfunction of the Hamiltonian, then it is time-independent, but if it is not, then the state function is given by (6.10) only at $t = 0$. In the following discussion, we shall consider the latter case.

[2] Particles which can be distinguished will be encountered in the next chapter.

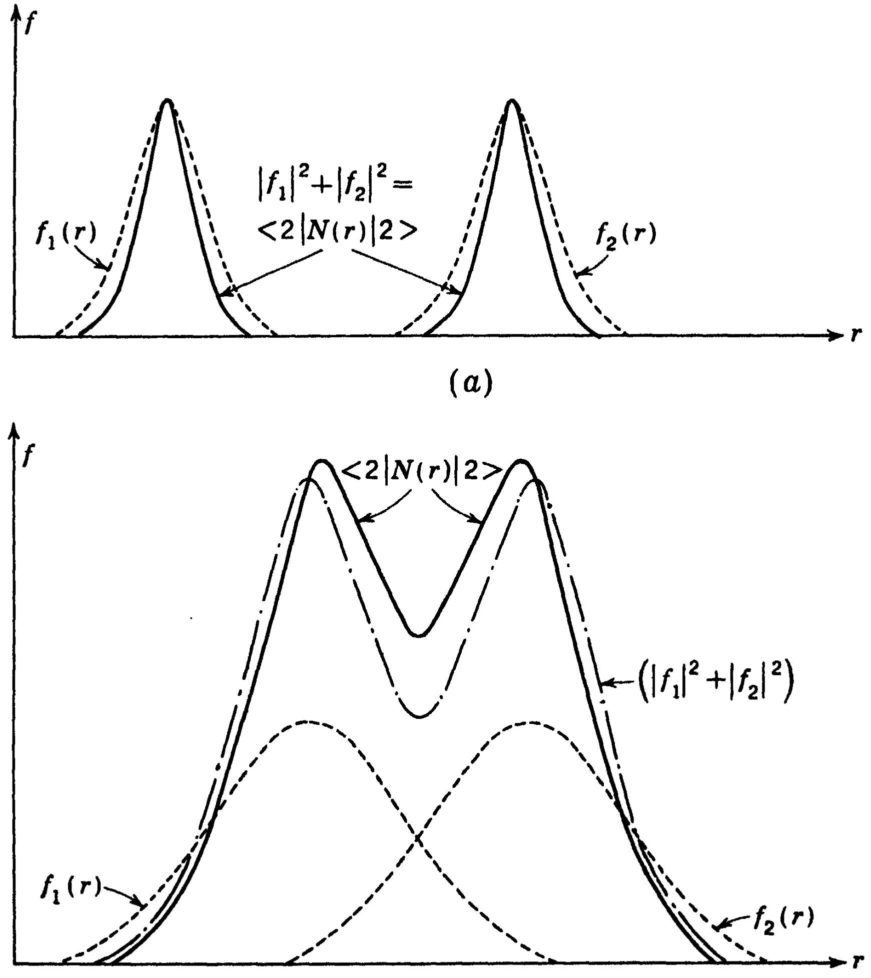
With this notation we find

$$\begin{aligned}\langle 2\,|N(\mathbf{r})|\,2\rangle &= \int d^3r_1\, d^3r_2\, d^3r_1'\, d^3r_2'\, f_1^*(\mathbf{r}_1) f_2^*(\mathbf{r}_2) f_1(\mathbf{r}_1') f_2(\mathbf{r}_2') \\ &\quad \times [1 + |(f_1,f_2)|^2]^{-1} \langle 0 \mid \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi^\dagger(\mathbf{r}_1')\psi^\dagger(\mathbf{r}_2') \mid 0\rangle \\ &= [|f_1(\mathbf{r})|^2 + |f_2(\mathbf{r})|^2 + (f_1,f_2) f_2^*(\mathbf{r}) f_1(\mathbf{r}) \\ &\quad + (f_2,f_1) f_1^*(\mathbf{r}) f_2(\mathbf{r})][1 + |(f_1,f_2)|^2]^{-1} \qquad (6.14)\end{aligned}$$

which is obtained by commuting all ψ's to the right and $\psi^\dagger$'s to the left and using $\psi \mid 0\rangle = \langle 0 \mid \psi^\dagger = 0$. We see that if the two wave functions are orthogonal, $(f_1,f_2) = 0$, the particle density is just the sum of the individual ones,

$$\langle 2 \mid N(\mathbf{r}) \mid 2\rangle = |f_1(\mathbf{r})|^2 + |f_2(\mathbf{r})|^2$$

as for independent particles and as shown in Fig. 6.1*a*. In particular,

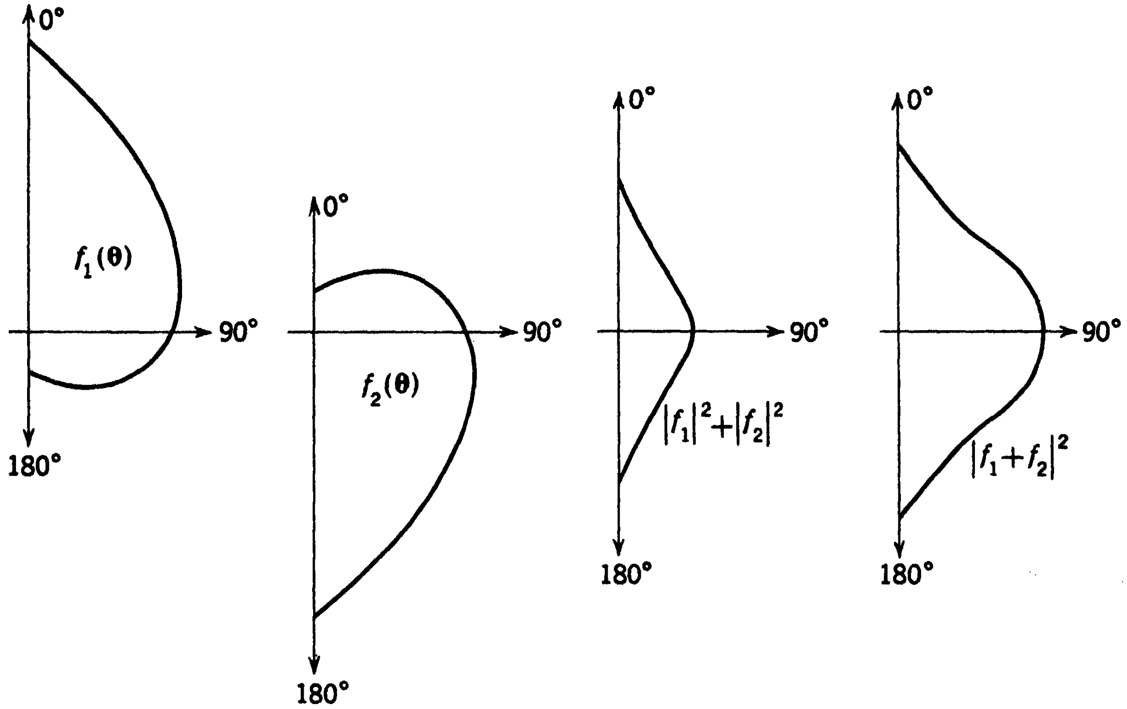


(*a*)

(*b*)

Fig. 6.1. Interference effects in the two-particle density distribution. Part *a* shows two noninterfering particles. Part *b* applies to the case of two particles which interfere. The density $|f_1|^2 + |f_2|^2$ is that which would apply for no interference. The actual density is $\langle 2\,|N(r)|\,2\rangle$ and shows that the particles tend to cluster in the interfering region.

this is the case in the classical limit of nonoverlapping wave packets, for which the particles can be identified by following their trajectories. However, if $(f_1,f_2) \neq 0$, there is an interference term, which decreases the density where the f's do not overlap, because of the denominator in (6.14), and hence increases it on the average in the overlapping region. This is demonstrated in Fig. 6.1*b*. Because of this property, bosons have a natural tendency to stick together. Hence, we see not only that a single particle can interfere with itself, which is the usual superposition principle in wave mechanics, but also that two identical particles can interfere with each other. This interference, which is another expression for the symmetry requirement of the wave function, does not occur between particles of different fields (see Chap. 7) and emphasizes that identical particles are just excitations of the same field. One important consequence occurs for the scattering of two identical particles. There the scattering intensity is not just the sum of the intensities for the two particles but includes an exchange term of the type displayed in (6.14). For example, if in the scattering of alpha particles by He^4 the center-of-mass-system amplitude $f_1(\theta)$ is peaked in the forward direction, then $f_2(\theta)$ must be peaked backward, but $|f|^2 = |f_1 + f_2|^2$ may then be anomalously large around 90°, as is shown in Fig. 6.2.



(*a*) (*b*) (*c*)

Fig. 6.2. Polar plots of the positive amplitudes $f_1(\theta)$ and $f_2(\theta)$ for hypothetical α-α scattering are shown in part *a*. The intensities without and with interference appear in parts *b* and *c*, respectively.

6.3. Many-particle States. Another unusual physical phenomenon directly related to the above is the fluctuation in the number of bosons in a volume $v \ll L^3$, where the latter volume contains a state with a definite number n of bosons. For independent particles the distribution of the number of particles in v satisfies a Poisson law, e.g., the probability of finding ν particles in v is

$$\eta_\nu = e^{-\bar{\nu}} \frac{(\bar{\nu})^\nu}{\nu!}$$

where $\bar{\nu}$ is the average number of particles in the volume,

$$\bar{\nu} = \sum_{\nu=0}^{\infty} \eta_\nu \nu \tag{6.15}$$

For a uniform distribution of particles in our normalization volume, $\bar{\nu}$ would be nv/L^3. For a Poisson distribution the fluctuation in the number of particles in v is

$$(\Delta\nu)^2 = \overline{\nu^2} - \bar{\nu}^2 = \sum_\nu \eta_\nu \nu^2 - \bar{\nu}^2 = \bar{\nu} \tag{6.16}$$

That is, for a normal uncorrelated distribution the square fluctuations in the local density are proportional to the density itself. This is the classical particle result.

Now, what do we obtain from a complete field theoretic treatment? It will be shown that even for bosons in orthogonal states (for which the particle densities are additive) the local fluctuations of the density of particles are not additive. To derive this, we first calculate $\bar{\nu}$ for the n-particle state which is represented by

$$| n \rangle = \int d^3r_1\, d^3r_2 \cdots d^3r_n\, f_1(\mathbf{r}_1) f_2(\mathbf{r}_2) \cdots f_n(\mathbf{r}_n) \psi^\dagger(\mathbf{r}_1) \psi^\dagger(\mathbf{r}_2) \cdots \psi^\dagger(\mathbf{r}_n) \,|\, 0 \rangle \tag{6.17}$$

with[1]

$$(f_i, f_j) = \delta_{ij} \tag{6.18}$$

With the same methods as before, this gives us

$$\begin{aligned} \bar{\nu} &= \langle n \,|\, N_v \,|\, n \rangle \\ &= \int_v d^3r\, \langle n \,|\, \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \,|\, n \rangle = \int_v d^3r \sum_{j=1}^{n} |f_j(\mathbf{r})|^2 \end{aligned} \tag{6.19}$$

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[1] We shall now take the f's to be orthogonal in order to separate the effect under consideration from the one discussed above.

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If each f is chosen to be a plane wave, $f_j = e^{i\mathbf{k}_j\cdot\mathbf{r}}/L^{\frac{3}{2}}$, we get

$$\bar{\nu} = \frac{nv}{L^3} \tag{6.20}$$

as was to be expected. For the expectation value

$$\begin{aligned} N_v^2 &= \int_v d^3r\, d^3r'\, N(\mathbf{r})N(\mathbf{r}') \\ &= N_v + \int_v d^3r\, d^3r'\, \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r})\psi(\mathbf{r}') \end{aligned} \tag{6.21}$$

we find in the usual way

$$\langle n \mid N_v^2 \mid n\rangle = \bar{\nu} + \sum_{i\neq k} \int_v d^3r\, d^3r'\, f_i^*(\mathbf{r})f_k^*(\mathbf{r}')[f_i(\mathbf{r})f_k(\mathbf{r}') + f_k(\mathbf{r})f_i(\mathbf{r}')] \tag{6.22}$$

and therefore

$$(\Delta N_v)^2 = \langle n \mid N_v^2 - \bar{N}_v^2 \mid n\rangle = \bar{\nu} + \sum_{i\neq k}\left|\int_v d^3r\, f_i^*(\mathbf{r})f_k(\mathbf{r})\right|^2 - \sum_i\left[\int_v d^3r\, |f_i(\mathbf{r})|^2\right]^2 \tag{6.23}$$

where $\qquad \bar{N}_v \equiv \langle n \,|\, N_v \,|\, n\rangle$

The first term in (6.23) is the result obtained earlier, Eq. (6.16), for an uncorrelated distribution; the two other terms represent the fluctuation due to the interference of the various particles. In the limit $v = L^3$ the second term vanishes by our assumption (6.18), and the last one is just n as expected, since we are then dealing with an eigenstate of N, for which $(\Delta N)^2 = 0$. For plane waves the last term is $n(v/L^3)^2 = \bar{\nu}v/L^3$ and vanishes for $\bar{\nu}$ = constant, $L^3 \to \infty$.

Let us study the fluctuations of the particle density in the two extreme cases in which the volume v is very much larger and very much smaller than the dimension of the volume in which the distribution functions f_j, representing the particles, are different from zero. In both cases we take v to be much smaller than L^3, since otherwise the fluctuations approach zero.[1] If the quanta are represented by wave packets with wavelength much smaller than $v^{\frac{1}{3}}$, as shown in Fig. 6.3*a*, then the second term drops out because of the orthogonality of f_j and $f_{k\neq j}$. The field then has properties of a purely noninterfering system and therefore no wavelike behavior. In the opposite extreme of long wavelength, $v \ll \lambda\!\!\!^{-3} = k_j^{-3}$, as shown in Fig. 6.3*b*, and for plane waves we obtain for the second term

$$\sum_{j\neq k}\left(\frac{v}{L^3}\right)^2 = \bar{\nu}^2$$

and the fluctuations are

$$(\Delta N_v)^2 = \bar{\nu}(\bar{\nu} + 1)$$

We note that for a low average density of particles, $\bar{\nu} \ll 1$, the particles behave like classical ones. To illustrate the significance of the added interference contribution (e.g., $\bar{\nu}^2$), we need a large density of particles, $\bar{\nu} \gg 1$. In this extreme the particles behave like a superposition of waves of equal amplitude and random phases,

---

[1] Remember that in this limit the third term on the right-hand side of (6.23) goes to zero.

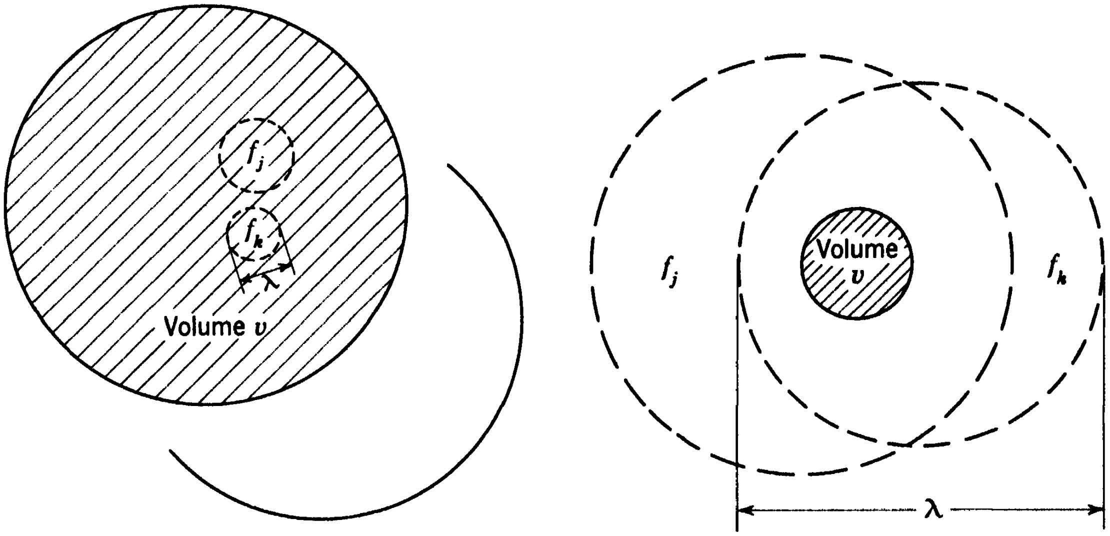
$$\sum_{j=1}^{n} e^{i\phi_j}$$

The intensity for the resulting wave is

$$I = \text{Re}\left(\sum_{j=1}^{n} e^{-i\phi_j} \sum_{k=1}^{n} e^{i\phi_k}\right) = n + 2\sum_{k>j} \cos(\phi_j - \phi_k) \tag{6.24}$$

and therefore
$$I^2 = [n + 2\sum_{k>j} \cos(\phi_j - \phi_k)]^2$$

In (6.24) Re means that the real part of the parenthesis is to be taken.



(a) (b)

Fig. 6.3. Fluctuation extremes. In part *a* the dimension of the packet representing the particle is much smaller than the volume v, and in part *b* it is much larger than v.

On averaging over the phases, we find, for $n \gg 1$,

$$\begin{aligned} \bar{I} = n \qquad \overline{I^2} &\approx 2n^2 \\ (\Delta I)^2 = \overline{I^2} - \bar{I}^2 &\approx n^2 \end{aligned} \tag{6.25}$$

in agreement with (6.23).[1] These large fluctuations stem from the natural tendency of bosons to cluster. This has, indeed, been observed in dense light beams where the counting rates of bosons do not follow a Poisson law.[2] Depending on whether there are few or many bosons within a wavelength, our system will exhibit either particle or wave properties.

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[1] Actually, our particles correspond to waves with amplitude $(v/L^3)^{\frac{1}{2}}$, rather than 1, which changes n into $\bar{\nu}$ in (6.25).

[2] See E. M. Purcell, *Nature*, **178**:1449 (1956).

CHAPTER 7

Internal Degrees of Freedom

7.1. Fields with Two Internal Degrees of Freedom. The quanta of the fields we have discussed behave like indistinguishable particles. To describe systems with distinguishable particles, it is necessary to introduce several fields, one for each of the kinds of particles. The particles may differ in such aspects as mass, spin, and spin direction or even in properties such as the charge, which are not connected to space time. It is the latter kind of distinction which will concern us in this chapter. Consider, for example, two hermitian Klein-Gordon fields $\phi_1 = \phi_1^\dagger$ and $\phi_2 = \phi_2^\dagger$ and take the Lagrangian density to be the sum

$$\begin{gathered}\mathscr{L} = \mathscr{L}(\phi_1) + \mathscr{L}(\phi_2) = \sum_{j=1}^{2} \tfrac{1}{2}[\dot{\phi}_j^2 - (\nabla\phi_j)^2 - m_j^2\phi_j^2] \\ [\phi_j(\mathbf{r},t), \dot{\phi}_l(\mathbf{r}',t)] = i\delta_{jl}\delta^3(\mathbf{r}-\mathbf{r}') \qquad (7.1) \\ [\phi_j(\mathbf{r},t), \phi_l(\mathbf{r}',t)] = [\dot{\phi}_j(\mathbf{r},t), \dot{\phi}_l(\mathbf{r}',t)] = 0\end{gathered}$$

This describes a system with two kinds of particles with different masses. Our previous considerations concerning the eigenstates of the various operators still apply, except that now each state has to be characterized by the number of particles of kinds 1 and 2. A two-particle state with one particle of each type, for instance, is given by

$$|\,2\rangle = \int d^3r_1\, d^3r_2\; f(\mathbf{r}_1,\mathbf{r}_2)\phi_1^{(-)}(\mathbf{r}_1)\phi_2^{(-)}(\mathbf{r}_2)\,|\,0\rangle \qquad (7.2)$$

but $f(\mathbf{r}_1,\mathbf{r}_2)$ need not now be symmetric, because

$$\phi_1^{(-)}(\mathbf{r}_1)\phi_2^{(-)}(\mathbf{r}_2) \neq \phi_1^{(-)}(\mathbf{r}_2)\phi_2^{(-)}(\mathbf{r}_1)$$

Hence particles of different fields do not obey Bose statistics, irrespective of whether or not they have the same space-time properties. Consequently they do not interfere with each other and do not show any anomalous fluctuations.

The mechanical model analogous to the introduction of two fields is a two-dimensional oscillator. The Lagrangian for this case is

$$L = \tfrac{1}{2}\sum_j (\dot{q}_j^2 - q_j^2\omega_j^2) \tag{7.3}$$

If the forces in the two directions are equal, e.g., $\omega_1 = \omega_2$, then a new nonergodic constant of the motion appears, owing to the rotational symmetry of the problem. This constant is the angular momentum around an axis perpendicular to 1 and 2. Exactly the same happens in the field theoretic case when the two masses in (7.1) are equal. The Lagrangian and the commutation rules are then invariant under the substitution

$$\begin{aligned} \phi_1' &= \phi_1 \cos\varphi + \phi_2 \sin\varphi \\ \phi_2' &= -\phi_1 \sin\varphi + \phi_2 \cos\varphi \end{aligned} \tag{7.4}$$

The relations (7.4) also express a rotational symmetry in a two-dimensional space, but this space has nothing to do with our space-time continuum. Nevertheless, the formal analogy suggests that ϕ_1 and ϕ_2 are the components of a two-dimensional vector field in an "internal space" with which there will be associated new constants of the motion arising from the rotational symmetry. These constants are called "isospin," in analogy to angular momentum for fields with three components. We shall discuss them presently. Like those stemming from the invariance of the space-time continuum, the constants are the generators of the infinitesimal transformations, and there are as many constants as there are parameters in the group which leaves L invariant. Since the invariance group (7.4) has one parameter, we have but one constant. To find this constant, we follow the usual pattern. Because (7.4) leaves the commutation rules (7.1) invariant, there must be a unitary operator U which connects[1] ϕ_j and ϕ_j':

$$\begin{aligned} U\phi_1U^{-1} &= \phi_1 \cos\varphi + \phi_2 \sin\varphi \\ U\phi_2U^{-1} &= -\phi_1 \sin\varphi + \phi_2 \cos\varphi \end{aligned} \tag{7.5}$$

---

[1] In field theory, unlike elementary quantum mechanics, it is not generally true that there is a unitary operator for every transformation which leaves the commutation rules invariant. However, we shall not get into trouble with these pathological nonequivalent representations in nonseparable Hilbert spaces. See A. S. Wightman and S. S. Schweber, *Phys. Rev.*, **98**:812 (1955).

For infinitesimal rotations, $\varphi \to \delta\varphi$, we put

$$U = 1 + i\delta\varphi Q \qquad U^{-1} = 1 - i\delta\varphi Q \qquad Q^\dagger = Q \tag{7.6}$$

and obtain

$$\begin{aligned} [Q,\phi_1] &= -i\phi_2 \\ [Q,\phi_2] &= i\phi_1 \end{aligned} \tag{7.7}$$

Q can, in fact, be constructed explicitly from the field operators and is the generalization of the expression for the angular momentum $p_1q_2 - p_2q_1$ in the mechanical model (7.3):

$$Q = \int d^3r\, [\dot{\phi}_1\phi_2 - \dot{\phi}_2\phi_1] \tag{7.8}$$

It is simple to verify by means of (7.1) that (7.6) and (7.7) are satisfied. Furthermore, like the Lagrangian, the operators H, $\mathbf{P}$, and $\mathbf{L}$ are of a unit quadratic form in the components ϕ_j and are therefore invariant under (7.5), or

$$[Q,H] = [Q,\mathbf{P}] = [Q,\mathbf{L}] = 0 \tag{7.9}$$

This tells us inversely that Q is invariant under spatial displacements and rotations and, in particular, is a constant in time. The latter can also be verified directly from the relation

$$\frac{d}{dt}(\dot{\phi}_1\phi_2 - \dot{\phi}_2\phi_1) + \nabla \cdot (\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) = 0 \tag{7.10}$$

which shows that Q can be reduced to a surface integral.

The same behavior is also found for the other constants, such as the energy, which are of the form of an infinite volume integral. The fact that their time derivatives vanish can be expressed in differential form by a continuity equation:

$$\frac{\partial Q(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

It follows from (7.10) that the local density $Q(\mathbf{r},t)$ of Q,

$$Q(\mathbf{r},t) = (\phi_2\dot{\phi}_1 - \phi_1\dot{\phi}_2) \tag{7.11}$$

and the current $\mathbf{j}$ defined by

$$\mathbf{j}(\mathbf{r},t) = (\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) \tag{7.12}$$

satisfy such a continuity equation. This suggests that they can be interpreted as the electric charge density and current of the field. This interpretation is sustained by their behavior under Lorentz transformations. The density $Q(\mathbf{r},t)$ and current $\mathbf{j}$ transform like a four-vector, as opposed, say, to the energy density which also satisfies a continuity equation. It is, therefore, possible to couple an electric field to $Q(\mathbf{r},t)$ and $\mathbf{j}$ in a Lorentz-invariant manner,

$$\mathscr{L}' = e[Q(\mathbf{r},t)V - \mathbf{j} \cdot \mathbf{A}]$$

where V is the electrostatic and $\mathbf{A}$ the vector potential. Of course, whether or not the particles actually have a charge e that is different from zero can only be discovered empirically.[1] Both cases exist in nature. For example, the K_0 and $\bar{K}_0$ particles do not couple to the electric field, whereas the π^+ and π^- mesons do. However, when particles are charged, they must be coupled to the electric field via a current of the form (7.12), because there is no other quantity that has the right transformation property and satisfies a continuity equation.

The eigenvalues of the charge operator Q can be inferred from the commutation rules (7.7), which can be written more compactly by means of a matrix

$$\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

as

$$[Q,\phi_j] = \sum_{l=1}^{2} (\tau_2)_{jl}\phi_l = (\tau_2\phi)_j$$

where matrix multiplication is implied in the last way of writing the right-hand side. Introducing those linear combinations of the fields ϕ_1 and ϕ_2 which diagonalize τ_2,

$$\phi_+ = \frac{\phi_1 + i\phi_2}{(2)^{\frac{1}{2}}} \qquad \phi_- = \frac{\phi_1 - i\phi_2}{(2)^{\frac{1}{2}}} \tag{7.13}$$

we find

$$[Q,\phi_+] = -\phi_+ \qquad [Q,\phi_-] = \phi_- \tag{7.14}$$

and

$$L = \int d^3r\,(\dot{\phi}_-\dot{\phi}_+ - \nabla\phi_-\cdot\nabla\phi_+ - m^2\phi_-\phi_+)$$

$$Q = i\int d^3r\,(\dot{\phi}_+\phi_- - \dot{\phi}_-\phi_+)$$

$$\pi_+ = \frac{\pi_1 - i\pi_2}{(2)^{\frac{1}{2}}} = \dot{\phi}_- \qquad \pi_- = \frac{\pi_1 + i\pi_2}{(2)^{\frac{1}{2}}} = \dot{\phi}_+$$

$$[\phi_+(\mathbf{r},t), \pi_+(\mathbf{r}',t)] = [\phi_-(\mathbf{r},t), \pi_-(\mathbf{r}',t)] = i\delta^3(\mathbf{r} - \mathbf{r}')$$

In terms of these fields the transformation (7.4) becomes a simple multiplication with a phase factor (gauge transformation of the first kind[2]):

$$\phi'_+ = e^{-i\varphi}\phi_+ \qquad \phi'_- = e^{i\varphi}\phi_-$$

The commutation rules (7.14) are of the standard form (2.7), but since Q is not positive definite, we may conclude that Q has both positive and negative integers as eigenvalues. We see that the eigenvalue problem for a generating operator, such as H, $\mathbf{P}$, $\mathbf{L}$, or Q, can always be solved by following the same procedure. The present case is particularly simple, since it amounts to diagonalizing a 2×2 matrix rather than a differential operator.

---

[1] For a single hermitian field we have only the number of particle density, and this does not satisfy a continuity equation, if there is an interaction. Hence the particles associated with this field must be neutral.

[2] This also shows that a single hermitian field can have no charge.

Since we do not wish the vacuum state to carry any charge, we require that

$$Q \mid 0\rangle = 0 \tag{7.15}$$

It should be noted that this requirement is satisfied without reordering the charge operator Q, according to the prescription (6.5). It follows from (7.14) that the one-particle states $\phi_+ \mid 0\rangle$ and $\phi_- \mid 0\rangle$ are eigenstates of the charge operator with eigenvalues -1 and $+1$, respectively, so that ϕ_- creates a positive particle and ϕ_+ a negative particle. Since Q commutes with $\mathbf{P}$ and H, we can also construct simultaneous eigenstates of these operators. If we define, with the notation (4.8),

$$\alpha_{\mathbf{k}} = \frac{a_{1\mathbf{k}} + i a_{2\mathbf{k}}}{(2)^{\frac{1}{2}}} \qquad \beta_{\mathbf{k}} = \frac{a_{1\mathbf{k}} - i a_{2\mathbf{k}}}{(2)^{\frac{1}{2}}} \tag{7.16}$$

and recognize that the transformation from a_1, a_2 to α, β is unitary, we obtain

$$\phi_+ = \sum_{\mathbf{k}} \frac{\alpha_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \beta_{\mathbf{k}}^{\dagger} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{(2\omega L^3)^{\frac{1}{2}}} \qquad \phi_- = \sum_{\mathbf{k}} \frac{\beta_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \alpha_{\mathbf{k}}^{\dagger} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{(2\omega L^3)^{\frac{1}{2}}}$$

$$[\alpha_{\mathbf{k}},\alpha_{\mathbf{k}}^{\dagger}] = [\beta_{\mathbf{k}},\beta_{\mathbf{k}}^{\dagger}] = 1 \qquad [\alpha_{\mathbf{k}},\alpha_{\mathbf{k}}] = [\beta_{\mathbf{k}},\beta_{\mathbf{k}}] = [\alpha_{\mathbf{k}},\beta_{\mathbf{k}}^{\dagger}] = [\alpha_{\mathbf{k}},\beta_{\mathbf{k}}] = \cdots = 0$$

and

$$\begin{aligned} H &= \sum_{\mathbf{k}} (\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}})\omega \\ \mathbf{P} &= \sum_{\mathbf{k}} (\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}})\mathbf{k} \\ Q &= \sum_{\mathbf{k}} (\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} - \beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}) \end{aligned} \tag{7.17}$$

From (7.17) it is apparent that the eigenvalues of the operators $\alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}}$ ($\beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}$) are the numbers of positive n_+ (negative n_-) particles with momentum $\mathbf{k}$ and energy ω.

It is interesting to observe that the commutation rules (7.7) or (7.14) also hold for $\phi_1(\mathbf{r})$ or $\phi_+(\mathbf{r})$ and the operator Q_v of the charge in a volume v, no matter how small,

$$Q_v = \int_v d^3r' \, Q(\mathbf{r}')$$

$$[Q_v,\phi_+(\mathbf{r})] = \begin{cases} -\phi_+(\mathbf{r}) & \text{if } \mathbf{r} \text{ is inside } v \\ 0 & \text{if } \mathbf{r} \text{ is outside } v \end{cases} \tag{7.18}$$

Therefore, the charge in an arbitrarily small volume also has integral eigenvalues, in contradistinction to the number-of-particles operator studied previously. The point nature of the charge quanta is also revealed by $[Q_v, Q_{v'}] = 0$ for arbitrary v and v' and is not in contradiction to our earlier findings that the particles which are eigenvalues of N have the size of a Compton wavelength. These particles have to be pictured as forming a fluctuating cloud of pointlike charge quanta[1] spread over a region of size m^{-3}.[¶] Indeed, $Q(\mathbf{r})$ does not commute with

$$N = N_+ + N_- = \sum_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}}$$

so that a state $\phi_-(\mathbf{r}) \mid \rangle$ defined such that $Q_v \mid \rangle = 0$, and for which, therefore,

$$Q_v\phi_-(\mathbf{r}) \mid 0\rangle = \begin{cases} \phi_-(\mathbf{r}) \mid 0 \rangle & \text{if } \mathbf{r} \text{ is inside } v \\ 0 & \text{if } \mathbf{r} \text{ is outside } v \end{cases}$$

will not be a one-particle state. In almost all the experiments we can perform at present, we barely have enough energy to excite the lowest eigenstates of H. Hence the particles we know empirically correspond to eigenstates of N and are complementary concepts to the charge quanta.

7.2. Three and More Degrees of Freedom. The rotational invariance of the two-dimensional charge space can be generalized; it then finds applications, for example, in pion physics, where we are concerned with three different particles—π^-, π^0, π^+—the description of which requires three fields. We shall consider the case of n different fields with equal masses first and shall then apply it to $n = 3$. Several fields of the same space-time properties seem to be realized in nature. For instance, the K mesons K^-, $\bar{K}_0$, K_0, K^+ correspond to four spin-zero fields. In general, we have

$$\mathscr{L} = \sum_{j=1}^{n} \tfrac{1}{2}[\dot{\phi}_j^2 - (\nabla\phi_j)^2 - m^2\phi_j^2]$$

$$[\phi_j(\mathbf{r}), \dot{\phi}_k(\mathbf{r}')]_{t=t'} = i\delta_{j,k}\delta^3(\mathbf{r} - \mathbf{r}') \tag{7.19}$$

$$\phi_j^\dagger = \phi_j \qquad [\phi_j(\mathbf{r}), \phi_k(\mathbf{r}')]_{t=t'} = [\dot{\phi}_j(\mathbf{r}), \dot{\phi}_k(\mathbf{r}')]_{t=t'} = 0$$

It is important to recognize that all terms in L must contribute with the same sign, in order to prevent negative contributions to the energy.[2]

---

[1] Of course, not all have the same sign of the charge.

[¶] This phenomenon is sometimes called *Zitterbewegung*. It was first found for the solutions of the Dirac equation. For a discussion within the framework of quantum field theory, see W. Thirring, "Principles of Quantum Electrodynamics," Academic Press, Inc., New York, 1958.

[2] Particles with negative energy spoil the dynamical stability of systems if an interaction is turned on.

The most general transformation which leaves (7.19) invariant is

$$U(\Lambda)\phi_i U^{-1}(\Lambda) = \sum_{j=1}^{n} \Lambda_{ij}\phi_j \qquad UU^\dagger = U^\dagger U = 1 \tag{7.20}$$

where Λ is time-independent and satisfies[1]

$$\Lambda\Lambda^\dagger = 1 \qquad \text{and} \qquad \Lambda^* = \Lambda \tag{7.21}$$

We therefore have rotational invariance in an n-dimensional euclidean space. For rotations through infinitesimal angles $\delta\varphi(r)$ about axes r, Λ can be written in the general form[2]

$$\Lambda_{kj} = \delta_{kj} + \sum_{r=1}^{N} \delta\varphi(r)\mathscr{T}^{(r)}_{kj} \tag{7.22}$$

where the $\delta\varphi(r)$ are infinitesimal (real) parameters. The number N of linearly independent matrices $\mathscr{T}^{(r)}$ will be determined below. In two dimensions the matrix $\mathscr{T}$ is $i\tau_2$. The restrictions imposed by (7.21), to order $\delta\varphi$, are

$$\mathscr{T}^{(r)}_{ij} = -\mathscr{T}^{(r)}_{ji} \qquad \mathscr{T}^{(r)*}_{ij} = \mathscr{T}^{(r)}_{ij} \tag{7.23}$$

Hence the $\mathscr{T}$ matrices must be real and antisymmetric. Thus the $\mathscr{T}$ have $n(n-1)/2$ linearly independent matrix elements, or we can choose that many linearly independent basic $n \times n$ matrices, satisfying (7.23), in terms of which the most general matrix satisfying this equation can be expressed. The number N of parameters which characterize the general $\mathscr{T}$ is therefore $n(n-1)/2$.

The operator U for the general infinitesimal Λ is again of the form

$$U = 1 + i\sum_{r=1}^{N} \delta\varphi(r)t^{(r)} \tag{7.24}$$

where $t^{(r)}$ must be hermitian, since U is to be unitary. The commutation relations for t which are obtained from (7.20) are

$$[t^{(r)},\phi_j] = -i\sum_{k=1}^{n} \mathscr{T}^{(r)}_{jk}\phi_k \qquad [t^{(r)},\dot{\phi}_j] = -i\sum_{k=1}^{n} \mathscr{T}^{(r)}_{jk}\dot{\phi}_k \tag{7.25}$$

and we recognize that (7.8) can be generalized to

$$t^{(r)} = \sum_{i,j=1}^{n} \int d^3r\, \dot{\phi}_i \mathscr{T}^{(r)}_{ij} \phi_j \tag{7.26}$$

Because the space-time constants are invariant under the transformation U, we have

$$[t^{(r)},H] = [t^{(r)},\mathbf{P}] = [t^{(r)},\mathbf{L}] = 0$$

---

[1] We use the following notation: T = transpose, $*$ = complex conjugate, $\dagger$ = hermitian conjugate.

[2] There are also constants associated with transformations not continuously connected with unity. They have interesting implications but are outside the scope of this book.

and each $t^{(r)}$ is thus a constant of the motion. We can also readily verify that there is a conserved current for each $t^{(r)}$. In two dimensions t is equal to Q.

In order to construct eigenstates of the operators $t^{(r)}$, we should first observe that, in general, $t^{(r)}$ and $t^{(s)}$ do not commute. We readily find from (7.26), or directly by means of (7.19), that the commutation relations among the operators $t^{(r)}$ can be calculated from those among the matrices $\mathscr{T}^{(r)}$,

$$\begin{aligned}[][t^{(r)},t^{(r')}] &= i\sum_{ijk}\int \dot{\phi}_i(\mathscr{T}^{(r)}_{ij}\mathscr{T}^{(r')}_{jk} - \mathscr{T}^{(r')}_{ij}\mathscr{T}^{(r)}_{jk})\phi_k\, d^3r \\ &= i\sum_{ij}\int d^3r\ \dot{\phi}_i[\mathscr{T}^{(r)},\mathscr{T}^{(r')}]_{ij}\phi_j \end{aligned} \tag{7.27}$$

where the second line of the equation has been written in matrix notation. Since the $[\mathscr{T}^{(r)},\mathscr{T}^{(r')}]$ are antisymmetric $n \times n$ matrices, they are linear combinations of the matrices

$$[\mathscr{T}^{(r)},\mathscr{T}^{(r')}] = \sum_{r''} C^{rr'}_{r''}\mathscr{T}^{(r'')} \tag{7.28}$$

Inserting into (7.27), we find that the operators t satisfy the same commutation rules as the matrices $\mathscr{T}$. Thus only those operators $t^{(r)}$ which have commuting $\mathscr{T}^{(r)}$ can be diagonalized simultaneously. Furthermore, we can conclude generally from (7.25) that the eigenvalues of the operators t are i times integral multiples of the eigenvalues of the matrices $\mathscr{T}$.

Finally, we shall discuss in more detail the case of $n = 3$, which is an appropriate description of π mesons.[1] A convenient choice of the antisymmetric matrices is then

$$\mathscr{T}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathscr{T}^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \mathscr{T}^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.29}$$

They satisfy the commutation rules

$$[\mathscr{T}^{(r)},\mathscr{T}^{(s)}] = \epsilon_{rst}\mathscr{T}^{(t)}$$

where ϵ_{rst} is the antisymmetric tensor. The three t operators corresponding to these matrices are called the isospin[2] and satisfy the same commutation relations as the angular-momentum $l = 1$ representation of the rotation operators

$$[t^{(r)},t^{(s)}] = i\epsilon_{rst}t^{(t)}$$

As for Q, we demand that

$$t^{(r)}\,|\,0\rangle = 0$$

---

[1] We use π mesons and pions interchangeably.

[2] In the literature the isospin is often referred to as the isotopic or isobaric spin.

It is consistent that the vacuum is a common eigenstate of all three (noncommuting) operators t, since it is also an eigenstate of the commutators with eigenvalue 0. In general, we can only have common eigenstates of one of the $t^{(r)}$ and of $t^2 = t^{(1)2} + t^{(2)2} + t^{(3)2}$. The distinction of one direction in isospace is physically obtained by the electric field which is coupled to one of the three conserved currents. By this coupling, one of the $t^{(r)}$, say $t^{(3)}$, obtains the significance of the electric charge. Its eigenstates are created by applying to the vacuum those linear combinations of the ϕ_j which diagonalize $t^{(3)}$. These are $(\phi_1 \pm i\phi_2)/2^{\frac{1}{2}}$, ϕ_3, the former belonging to the eigenvalue ± 1 (describing $\pi^\pm$) and the latter to eigenvalue 0 (describing π^0). The common eigenstates of t^2 and $t^{(3)}$ can be constructed analogously to the ones for angular momenta. Calling $t'(t'+1)$ the eigenvalue of t^2, we see that the vacuum is characterized by $t' = 0$ and the one-meson states by $t' = 1$. Two mesons can have $t' = 0, 1, 2$. The state $t' = 0$ is invariant under rotations in isospace and hence proportional to the scalar product

$$
\begin{aligned}
|\, n = 2, t' = 0\rangle &= \sum_{i=1}^{3} \int d^3r_1\, d^3r_2\, \phi_i^{(-)}(\mathbf{r}_1)\phi_i^{(-)}(\mathbf{r}_2)F_0(\mathbf{r}_1,\mathbf{r}_2)\,|\,0\rangle \\
&= \int d^3r_1\, d^3r_2[\phi_+^{(-)}(\mathbf{r}_1)\phi_-^{(-)}(\mathbf{r}_2) + \phi_-^{(-)}(\mathbf{r}_1)\phi_+^{(-)}(\mathbf{r}_2) \\
&\quad + \phi_3^{(-)}(\mathbf{r}_1)\phi_3^{(-)}(\mathbf{r}_2)]F_0(\mathbf{r}_1,\mathbf{r}_2)\,|\,0\rangle
\end{aligned}
$$

In terms of charged particles, this is of the form

$$|\, - \; +\rangle + |\, + \; -\rangle + |\, 0\, 0\rangle$$

and hence each of the two particles has equal probabilities of being positive, neutral, or negative.[1] The state with $t = 1$ must transform like a vector under rotations in isospace and hence can be represented by the vector product

$$|\, n = 2, t' = 1\rangle = \sum_{i,j} \int d^3r_1\, d^3r_2\, \epsilon_{ijk}\phi_i^{(-)}(\mathbf{r}_1)\phi_j^{(-)}(\mathbf{r}_2)F_1(\mathbf{r}_1,\mathbf{r}_2)\,|\,0\rangle$$

In particular, the $t_3 = 0$ state ($k = 3$) is of the form $|\, + \; -\rangle - |\, - \; +\rangle$. Finally, the $t = 2$ states transform like a symmetric traceless tensor,

$$
\begin{aligned}
|\, n = 2, t' = 2\rangle = \int d^3r_1\, d^3r_2\, [&\phi_i^{(-)}(\mathbf{r}_1)\phi_j^{(-)}(\mathbf{r}_2) + \phi_j^{(-)}(\mathbf{r}_1)\phi_i^{(-)}(\mathbf{r}_2) \\
&-\tfrac{2}{3}\delta_{ij}\phi_i^{(-)}(\mathbf{r}_1)\phi_j^{(-)}(\mathbf{r}_2)]F_2(\mathbf{r}_1,\mathbf{r}_2)\,|\,0\rangle
\end{aligned}
$$

---

[1] The notation $\phi_+^{(-)}$, etc., is a straightforward generalization of (5.18),

$$\phi_+^{(-)} = \sum_{\mathbf{k}} \frac{\beta_{\mathbf{k}}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{(2\omega L^3)^{\frac{1}{2}}}$$

Note that, since the internal-space rotational symmetry is independent of space-time properties, it follows that $F_0(\mathbf{r}_1,\mathbf{r}_2)$ is independent of $i = 1, 2, 3$.

These states can also be formed by standard angular-momentum addition formulas.[1] Note that the isospin parts of the $t = 0, 2$ states are even and those of the $t = 1$ states are odd under exchange of the two particles. Hence $F(\mathbf{r}_1,\mathbf{r}_2)$ must be even in the former and odd in the latter case under exchange of $\mathbf{r}_1$ and $\mathbf{r}_2$,¶ a result which is important for the pion cloud around the nucleon.

In summary, the different particles found in nature can easily be fitted into the framework of field theory. The more specific predictions of a symmetry isomorphic to the three-dimensional euclidean group for the three pions will be important in the last part of the book. There we shall find that this invariance is not destroyed by the strongest interactions found in nature, so that the above considerations are an important tool in pion physics.

[1] See, e.g., E. U. Condon and G. H. Shortley, "The Theory of Atomic Spectra," chap. III, Cambridge University Press, New York, 1953.

¶ Parts of F which have the wrong symmetry cancel out, as was explained above.

Part Two

SOLUBLE INTERACTIONS

CHAPTER 8

General Orientation

8.1. Field Equations. So far we have been considering free fields. We shall now turn to the more interesting problem involving an additional mechanism which can create, absorb, and scatter the field particles. In this part of the book we shall describe several such interactions which are tractable and can be solved exactly. Unfortunately, they are rather remote from physical reality and bear only a faint resemblance to what one finds in nature. Nevertheless, their study is of more than academic interest, since they teach us what might happen in the more realistic cases which cannot be analyzed in detail, such as the pion-nucleon interaction, which we shall study in the last part of the book.

As a first example, let us consider the case of a simple field source $\rho(\mathbf{r},t)$ which is a prescribed function of space and time:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(\mathbf{r},t) = \rho(\mathbf{r},t) \tag{8.1}$$

The simplest mechanical analogue to this kind of problem is an external force $f(t)$ applied to a harmonic oscillator. The equation of motion is then

$$\ddot{q} + \omega^2 q = f(t) \tag{8.2}$$

As in our previous considerations, we shall first orient ourselves about the classical solutions of such an equation and shall later consider the quantum aspects of the problem. As we shall see, Eq. (8.1) can be solved with the aid of the Green's function, like the well-known one-dimensional case (8.2).

To this end, we have to realize that if the source is the sum of several parts, then, owing to the linearity of the equations, the solution is the sum of the solutions corresponding to the individual parts. Hence we only need to find a solution for a point source

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)G(\mathbf{r},t) = \delta(t)\delta^3(\mathbf{r}) \tag{8.3}$$

and can then build up a solution of (8.1) by superposition:

$$\phi(\mathbf{r},t) = \int dt'\, d^3r'\, G(\mathbf{r} - \mathbf{r}', t - t')\rho(\mathbf{r}',t') \tag{8.4}$$

It is readily seen, with the aid of (8.3), that this field actually satisfies (8.1). A solution of (8.3) can be obtained by an expansion in terms of the eigenfunctions of the differential operator $\partial^2/\partial t^2 - \nabla^2$, that is, by introducing a Fourier transformation:

$$G(\mathbf{r},t) = \mathop{\mathsf{S}}_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{dK_0}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - K_0 t)} g(K_0,\mathbf{k}) \tag{8.5}$$

$$\delta^3(\mathbf{r})\,\delta(t) = \mathop{\mathsf{S}}_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{dK_0}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - K_0 t)}$$

Substitution into (8.3) gives

$$\mathop{\mathsf{S}}_{\mathbf{k}} \int_{-\infty}^{\infty} \frac{dK_0}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - K_0 t)}[(K_0^2 - \mathbf{k}^2 - m^2)g + 1] = 0 \tag{8.6}$$

or

$$g = \frac{1}{m^2 + k^2 - K_0^2} \tag{8.7}$$

We note that the integrand in (8.5) has two poles on the path of integration, at $K_0 = \omega \equiv (k^2 + m^2)^{\frac{1}{2}}$ and at $K_0 = -\omega$. Without specification of the path of integration at the singularities, there is an ambiguity in our expressions for the Green's function. On integrating along different paths, we get results which differ by the residues at the singularities. This is the well-known fact that the solution of a linear inhomogeneous equation is not unique, since a solution of the homogeneous equation can always be added to it. In fact, the contributions from the residues are of the form $e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, which is just the solution of the homogeneous equation. To obtain a unique solution, it is necessary to impose boundary conditions. The ones of special importance for the problems that we shall be concerned with are characterized by the paths of integration in the complex K_0 plane shown in Fig. 8.1. We shall denote these particular Green's functions by Δ^{ret} and Δ^{adv}. Their significance can be seen by studying the K_0 integral in the complex K_0 plane. For t larger (less) than zero, the factor $e^{-iK_0 t}$ increases (decreases) exponentially in the upper half plane and decreases (increases) in the lower half. Closing the path of integration by adding an infinite semicircle in the upper or lower half plane, we see that $\Delta^{\mathrm{ret}} = 0$ for $t < 0$ and that $\Delta^{\mathrm{adv}} = 0$ for $t > 0$.[¶] The complete Green's function Δ^{ret} is given by[1]

$$\Delta^{\mathrm{ret}}(\mathbf{r},t) = \begin{cases} \mathop{\mathrm{S}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \dfrac{\sin \omega t}{\omega} & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{8.8}$$

and the two solutions Δ^{ret} and Δ^{adv} are related by

$$\Delta^{\mathrm{ret}}(\mathbf{r},t) = \Delta^{\mathrm{adv}}(\mathbf{r},-t) \tag{8.9}$$

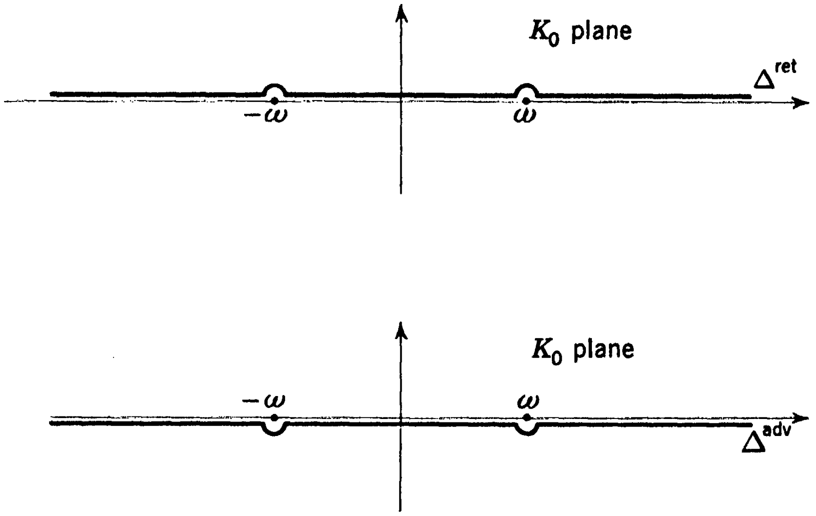

Fig. 8.1. Green's function contours corresponding to two different boundary conditions.

With the aid of these Green's functions, we can write the general solution of (8.1) in the form

$$\begin{aligned} \phi(\mathbf{r},t) &= \phi^{\mathrm{in}}(\mathbf{r},t) + \int dt'\, d^3r'\, \Delta^{\mathrm{ret}}(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t') \\ &= \phi^{\mathrm{out}}(\mathbf{r},t) + \int dt'\, d^3r'\, \Delta^{\mathrm{adv}}(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t') \end{aligned} \tag{8.10}$$

Here ϕ^{in} and ϕ^{out} are solutions of the homogeneous equation. Their physical meaning can be seen most easily if the source $\rho(\mathbf{r},t)$ differs from zero only in a finite space-time region, bounded by the times t_1 and t_2. It follows from the above properties of the Green's function that in this case ϕ coincides with ϕ^{in} for $t < t_1$ and with ϕ^{out} for $t > t_2$. Hence ϕ^{in}

[¶] One can see from relativistic invariance that $\Delta^{\mathrm{ret}} = 0$ even for $t < r$.

[1] Both Δ^{ret} and Δ^{adv} can be worked out to be Hankel functions of $(r^2 - t^2)$. This is most conveniently done by relating Δ^{ret} to Δ^{+}; see, e.g., W. Thirring, "Principles of Quantum Electrodynamics," Academic Press, Inc., New York, 1958. However, we shall not need these expressions.

represents the field which was present before the source was switched on, and ϕ^{out} is the field which is left over after the source has been turned off, as shown in Fig. 8.2. In many discussions, only the term with Δ^{ret} in (8.10) is used, which means that one imposes the boundary condition that $\phi = \phi^{\text{in}}$ at $t = -\infty$. Although there is no fundamental reason why Δ^{ret} should be better than Δ^{adv}, the former is used more frequently because the initial experimental conditions at $t = -\infty$ are more easily prepared than specific final conditions at $t = +\infty$.

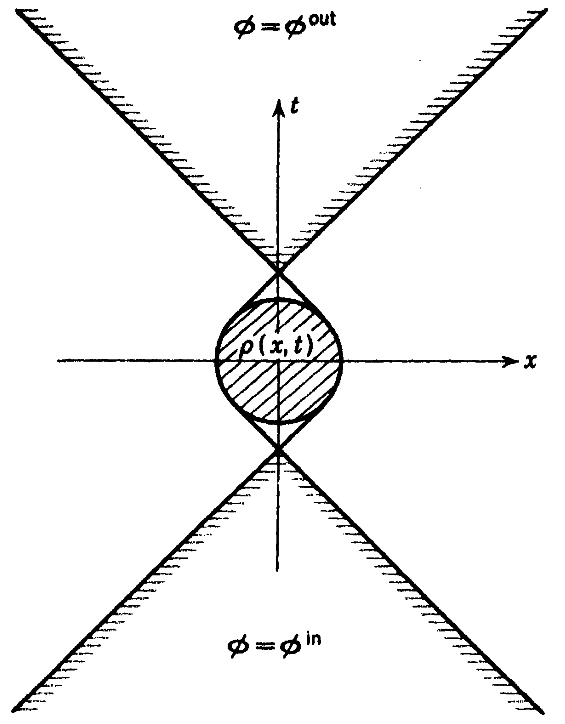

Fig. 8.2. Regions about a source $\rho(x,t)$ in which ϕ corresponds to ϕ^{in} and ϕ^{out}.

Another kind of problem which is of interest, and which will be studied later on, is characterized by the equation[1]

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(\mathbf{r},t) = \int d^3r'\, V(\mathbf{r},\mathbf{r}',t)\phi(\mathbf{r}',t) \quad (8.11)$$

$V(\mathbf{r},\mathbf{r}',t)$ represents a generalized potential that acts on the field; for coupled oscillators it corresponds to a more complicated coupling than one that is just between nearest neighbors. It is only possible to solve (8.11) explicitly for particular forms of V, but in any case the equation can be rewritten in integral form with the aid of the Green's functions:

$$\begin{aligned}\phi(\mathbf{r},t) &= \phi^{\text{in}}(\mathbf{r},t) + \int dt'\, d^3r'\, d^3r''\, \Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')V(\mathbf{r}',\mathbf{r}'',t')\phi(\mathbf{r}'',t') \\ &= \phi^{\text{out}}(\mathbf{r},t) + \int dt'\, d^3r'\, d^3r''\, \Delta^{\text{adv}}(\mathbf{r}-\mathbf{r}', t-t')V(\mathbf{r}',\mathbf{r}'',t')\phi(\mathbf{r}'',t) \quad (8.12)\end{aligned}$$

If V tends to zero for $t \to \pm\infty$, then ϕ^{in} and ϕ^{out} as defined by (8.12) have the same physical significance as in the previous example.

Both kinds of problems are encountered in many branches of physics, although usually in somewhat more complicated form. A typical feature of such systems is that the energy and momentum of the field alone will no longer be a constant. This stems from the fact that

---

[1] This kind of equation is familiar from ordinary wave mechanics, except that one usually deals with time-independent and spatially localized potentials, $V(\mathbf{r},\mathbf{r}',t) = V_0(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{r}')$.

$\partial L/\partial t$ and $\partial L/\partial r$ will, in general, not be zero, since ρ or V may depend explicitly on t and r and since L now contains an additional term with ρ or V. In fact, not even the number of field particles will be conserved, since (8.1) and (8.11) give

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\int d^3r\, i(\phi^{(-)}\dot{\phi}^{(+)} - \dot{\phi}^{(-)}\phi^{(+)}) \neq 0$$

The observables remain constant only in special circumstances. The angular momentum, for instance, is constant for a spherically symmetric source.

8.2. Quantization. In order to discuss the quantum theory of our two cases, we follow the standard procedure. The Lagrangians of the systems differ from the Lagrangian of the free fields L_0 by the addition of terms

$$L' = \int d^3r\, \rho(\mathbf{r},t)\phi(\mathbf{r},t) \tag{8.13}$$

and

$$L' = \tfrac{1}{2}\int d^3r\, d^3r'\, \phi(\mathbf{r},t)V(\mathbf{r},\mathbf{r}',t)\phi(\mathbf{r}',t) \tag{8.14}$$

respectively. The hermiticity of L' implies $V(\mathbf{r},\mathbf{r}',t) = V^*(\mathbf{r},\mathbf{r}',t)$, and we take $V(\mathbf{r},\mathbf{r}',t) = V(\mathbf{r}',\mathbf{r},t)$. Since L' does not contain $\dot{\phi}$ in either case, the canonical commutation relations remain

$$[\phi(\mathbf{r},t), \dot{\phi}(\mathbf{r}',t)] = i\delta^3(\mathbf{r} - \mathbf{r}') \tag{8.15}$$

To get some information about the commutation properties of ϕ^{in} and ϕ^{out}, we consider the limits $t \to \pm\infty$ where ρ and V approach zero. Since ϕ then coincides with ϕ^{out} and ϕ^{in}, (8.15) implies

$$\begin{aligned} [\phi^{\text{in}}(\mathbf{r},-\infty), \dot{\phi}^{\text{in}}(\mathbf{r}',-\infty)] &= i\delta^3(\mathbf{r} - \mathbf{r}') \\ [\phi^{\text{out}}(\mathbf{r},\infty), \dot{\phi}^{\text{out}}(\mathbf{r}',\infty)] &= i\delta^3(\mathbf{r} - \mathbf{r}') \end{aligned} \tag{8.16}$$

The operators ϕ^{in} and ϕ^{out} obey the homogeneous field equations and can therefore be expressed in terms of time-independent operators A_k and B_k in the familiar form

$$\begin{aligned} \phi^{\text{in}} &= \sum_{\mathbf{k}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}A_{\mathbf{k}} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}A_{\mathbf{k}}^{\dagger}}{(2\omega L^3)^{\frac{1}{2}}} \\ \phi^{\text{out}} &= \sum_{\mathbf{k}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}B_{\mathbf{k}} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}B_{\mathbf{k}}^{\dagger}}{(2\omega L^3)^{\frac{1}{2}}} \end{aligned} \tag{8.17}$$

Our development for free fields provides, therefore, a basis for the problem with interactions.

From (8.16) we infer that the A, ($A^\dagger$) and B, ($B^\dagger$) obey the usual commutation rules of destruction (creation) operators. This tells us, in turn, that (8.16) holds for all times and not only in the limits $t \to \pm\infty$. The compatibility of (8.15) and (8.16) is evident for the first case where ϕ and $\phi^{\substack{\text{in}\\ \text{out}}}$ differ only by an ordinary number. In general, however, the equivalence of the two commutation rules will not be trivial and will have some important implications. At a certain time t, one can, of course, satisfy the commutation relations of the local field operators $\phi(\mathbf{r},t)$ and $\dot{\phi}(\mathbf{r},t)$ in the customary manner:

$$\begin{aligned} \phi(\mathbf{r},t) &= \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}(t) + e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_{\mathbf{k}}(t)}{(2\omega L^3)^{\frac{1}{2}}} \\ \pi(\mathbf{r},t) = \dot{\phi}(\mathbf{r},t) &= -\sum_{\mathbf{k}} i\left(\frac{\omega}{2L^3}\right)^{\frac{1}{2}}[a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - e^{-i\mathbf{k}\cdot\mathbf{r}}a^\dagger_{\mathbf{k}}(t)] \end{aligned} \tag{8.18}$$

where the operators $a_{\mathbf{k}}(t)$ satisfy the usual commutation relations at equal times,

$$[a_{\mathbf{k}}(t), a^\dagger_{\mathbf{k}'}(t)] = \delta^3_{\mathbf{k},\mathbf{k}'} \tag{8.19}$$

It is important to note, however, that (8.18) and (8.19) do not imply that the time dependence of the operators $a_{\mathbf{k}}$ is that of the free fields. This will, in fact, not be the case, except in the limits of $t \to \pm\infty$.

The physical interpretation of our systems in quantum theory is developed along the same lines as in the classical field case. Since for $t \to -\infty$, ϕ coincides with ϕ^{in}, the quanta which are created and destroyed by $A^\dagger$ and A are those particles which were present before the source was turned on. In particular, we can define the number operator $N^{\text{in}}_{\mathbf{k}}$ of incoming particles with momentum $\mathbf{k}$ as

$$N^{\text{in}}_{\mathbf{k}} = A^\dagger_{\mathbf{k}}A_{\mathbf{k}} \tag{8.20}$$

Because ϕ^{in} obeys the free-field equation, $N^{\text{in}}_{\mathbf{k}}$ is a constant, and its eigenstates represent a situation wherein a definite number of particles with momentum $\mathbf{k}$ are initially ($t \to -\infty$) present. The same considerations apply to ϕ^{out} and $N^{\text{out}}_{\mathbf{k}}$, which correspond to the actual situation after the source has been turned off. At any time, and in particular when the source is switched on, the field is represented by the operators $a_{\mathbf{k}}(t)$ and the number of particles by

$$N_{\mathbf{k}}(t) = a^\dagger_{\mathbf{k}}(t)a_{\mathbf{k}}(t)$$

The latter is not a constant and will differ from $N^{\text{in}}_{\mathbf{k}}$ once the source is turned on. Therefore, an eigenstate of $N^{\text{in}}_{\mathbf{k}}$ has a fluctuating number of $N_{\mathbf{k}}(t)$ particles. Even a state with no incoming particles will not be an eigenstate of the operator $N_{\mathbf{k}}(t)$.

The particles represented by $N_{\mathbf{k}}(t)$ are usually called bare particles, and generally $N_{\mathbf{k}}(t)$ represents both real and virtual particles. The eigenstates of $N_{\mathbf{k}}(t)$ which are also eigenstates of $H_0(t)$,

$$H_0(t) = \tfrac{1}{2}\int d^3r\,[\pi^2 + (\nabla\phi)^2 + m^2\phi^2] = \sum_{\mathbf{k}} \omega a_{\mathbf{k}}^\dagger(t) a_{\mathbf{k}}(t) \qquad (8.21)$$

are called the bare states. Since $N_{\mathbf{k}}(t)$ and $H_0(t)$ are time-dependent, corresponding bare states at different times will be different.[1] The eigenstates of $N_{\mathbf{k}}^{\text{in}}$ which are eigenstates of H are called physical states, and the corresponding particles are real particles. That the states generated by $A_{\mathbf{k}}^\dagger(t)$ are eigenstates of the total Hamiltonian follows simply from the fact that the time dependence of the $A_{\mathbf{k}}(t)$ is that for free fields. We shall see this more explicitly in the following chapters, where we shall express H in terms of the operators A_k. We shall find that H is simply $\sum_{\mathbf{k}} A_{\mathbf{k}}^\dagger A_{\mathbf{k}} \omega$ plus a c number. The bare states are hard to prepare experimentally, since usually sufficient energy is not available to excite more than the lowest few states of the system. For instance, a bare vacuum state[2] $|\,0)$ defined by $a_{\mathbf{k}}(t)\,|\,0) = 0$ corresponds to one for which the dress of virtual particles from the source is removed at time t. This costs a lot of energy, since $|\,0)$ contains an admixture of highly excited physical states.

In our cases the situation is relatively simple, since we only have the source which is capable of emitting particles. In a theory with nonlinear terms in the field equations, as in relativistic quantum electrodynamics, each particle acts as source for the other particles. There each physical particle is a mixture of all sorts of bare particles. In our theories the physical states are the source plus a certain number of incoming (or outgoing) particles. These consist of the bare source plus a certain configuration of bare particles.

One may ask to what extent the virtual particles possess physical significance. Certainly the particles we see in cloud or bubble chambers are always the physical particles. However, the virtual particles do exist, inasmuch as they lead to observable effects. We shall see that they contribute to the energy and charge distribution of the system. Furthermore, as we shall explicitly see later on, the virtual particles present at the time t can be made real by suddenly switching off the source at this time. In this case $\phi(\mathbf{r}, t + \delta t)$ will turn out to be identical with $\phi^{\text{out}}(\mathbf{r}, t + \delta t)$ and with $\phi(\mathbf{r},t)$ ($\dot{\phi}$ stays finite). Since N^{out} is constant, the particles created remain for all later time and represent the particles detected afterward. The virtual particles are thus those which would be left over if the source should suddenly be turned off. In practice this can happen to some extent,[1] e.g., through the annihilation of a nucleon by an antinucleon or through a very fast collision between two nucleons.[2] The mesons produced in such events are just the virtual particles in the meson cloud of the nucleon, which suddenly find themselves without their source. The field ϕ^{out} will in general differ from ϕ^{in}. Therefore, the number of particles and their energy and momentum at $t \to \infty$ will differ from those at $t \to -\infty$, and both will differ from corresponding quantities at other times.

---

[1] Remember that we are working in the Heisenberg representation, where all states are constant and the time dependence is put into the operators.

[2] To avoid the crowding of labels, we denote bare states by $|\,)$ and physical states by $|\,>$.

8.3. Scattering and Wave Matrix. We found that for coupled fields the concept of a particle requires some qualifications. In the table below we summarize the different sets of orthogonal states which are associated with the various kinds of particles:

$$\begin{aligned} N^{\text{in}}\,|\,\text{in},0\rangle &= \sum_{\mathbf{k}} A_{\mathbf{k}}^{\dagger}A_{\mathbf{k}}\,|\,\text{in},0\rangle = 0 \\ |\,\text{in},n_{\mathbf{k}}\rangle &= \frac{1}{(n_{\mathbf{k}}!)^{\frac{1}{2}}}(A_{\mathbf{k}}^{\dagger})^{n_{\mathbf{k}}}\,|\,\text{in},0\rangle \\ H\,|\,\text{in},n_{\mathbf{k}}\rangle &= E_{\mathbf{k}}\,|\,\text{in},n_{\mathbf{k}}\rangle \\ [H,A_{\mathbf{k}}] &= -\omega A_{\mathbf{k}} \end{aligned}$$

These are physical states. They represent the dressed source plus n incoming particles.

$$\begin{aligned} N^{\text{out}}\,|\,\text{out},0\rangle &= \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger}B_{\mathbf{k}}\,|\,\text{out},0\rangle = 0 \\ |\,\text{out},n_{\mathbf{k}}\rangle &= \frac{1}{(n_{\mathbf{k}}!)^{\frac{1}{2}}}(B_{\mathbf{k}}^{\dagger})^{n_{\mathbf{k}}}\,|\,\text{out},0\rangle \\ H\,|\,\text{out},n_{\mathbf{k}}\rangle &= E_{\mathbf{k}}'\,|\,\text{out},n_{\mathbf{k}}\rangle \\ [H,B_{\mathbf{k}}] &= -\omega B_{\mathbf{k}} \end{aligned}$$

These are physical states. They represent the dressed source plus n outgoing particles.

$$\begin{aligned} N(t)\,|\,0) &= 0 \\ |\,n_{\mathbf{k}}) &= \frac{1}{(n_{\mathbf{k}}!)^{\frac{1}{2}}}(a_{\mathbf{k}}^{\dagger})^{n_{\mathbf{k}}}\,|\,0) \\ H_0(t)\,|\,0) &= 0 \\ H_0(t)\,|\,n_{\mathbf{k}}) &= \omega n_{\mathbf{k}}(t)\,|\,n_{\mathbf{k}}) \\ [H_0,a_{\mathbf{k}}] &= -\omega a_{\mathbf{k}} \end{aligned}$$

These are bare states. They represent the bare source plus $n_{\mathbf{k}}$ (real or virtual) particles.

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[1] It would have to be carried out infinitely fast to produce all excited physical states.

[2] See H. W. Lewis, J. R. Oppenheimer, and S. Wouthuysen, *Phys. Rev.*, **73**:127 (1948); E. M. Henley and T. D. Lee, *Phys. Rev.*, **101**:1536 (1955); Z. Koba and G. Takeda, *Progr. Theoret. Phys. (Kyoto)*, **19**:269 (1958).

In certain circumstances the "in" and "out" states do not form a complete set—e.g., when the source is strong enough to bind particles. In this case one has to augment the in and out states with the bound states to obtain a complete set. We shall discuss this in detail when the case arises but shall assume in the discussion below that all sets are complete. Then they are related to one another by unitary matrices. The elements of these matrices can be defined as the products of the states of one set and those of another. Equivalently, they can be defined as the elements of a matrix which transforms the generating operator of one set into that of the other. For instance, the connection between the in and out states is established by the so-called "S matrix," or "scattering matrix," which plays a crucial role in modern field theory. We can define it as the unitary matrix which transforms the $A_{\mathbf{k}}(t)$ into the $B_{\mathbf{k}}(t)$:

$$\begin{aligned} B_{\mathbf{k}}(t) &= S^{-1}A_{\mathbf{k}}(t)S \\ S^{\dagger}S &= SS^{\dagger} = 1 \end{aligned} \tag{8.22}$$

The existence of such a matrix is inferred by the usual remark that the $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ satisfy identical commutation relations. Furthermore, they have the same time dependence, so that S is time-independent. From (8.22) we infer

$$A_{\mathbf{k}}S \mid 0, \text{out}\rangle = 0 \qquad S \mid 0, \text{out}\rangle = \mid 0, \text{in}\rangle \tag{8.23}$$

Hence an equivalent definition of the elements of S is[1]

$$\begin{aligned} S_{\mathbf{k}_1',\ldots,\mathbf{k}_n',\mathbf{k}_1,\ldots,\mathbf{k}_n} &= \langle \text{in}, \mathbf{k}_1', \ldots, \mathbf{k}_n' \mid S \mid \text{in}, \mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \\ &= \langle \text{in}, 0 \mid A_{\mathbf{k}_1'}, \ldots, A_{\mathbf{k}_n'}S \mid \text{in}, \mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \\ &= \langle \text{out}, 0 \mid B_{\mathbf{k}_1'}, \ldots, B_{\mathbf{k}_n'} \mid \text{in}, \mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \\ &= \langle \text{out}, \mathbf{k}_1', \ldots, \mathbf{k}_n' \mid \text{in}, \mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \end{aligned} \tag{8.24}$$

One of the important features of the S matrix is its relation to the scattering cross section, which is sketched below. In systems wherein the total energy is conserved, the S matrix only connects states with equal energy. It is conventional to write the matrix element of S between an initial state with energy ω_i and a final state with energy ω_f in the form

$$S_{fi} = \delta_{fi} - 2\pi i\delta(\omega_i - \omega_f)T_{fi} \tag{8.25}$$

From this relation, the probability that a final state $f \neq i$ develops from the initial one is found to be

$$\eta_{fi} = 4\pi^2 \left|\delta(\omega_i - \omega_f)T_{fi}\right|^2 \tag{8.26}$$

---

[1] We assume that all the values of $\mathbf{k}$ are different; otherwise the normalization differs from that shown.

A rough manner of arguing away the unpleasant δ^2 is to observe that it is connected with the infinite time of interaction in eigenstates of the energy. The $\delta(\omega_i - \omega_f)$ has its origin in an expression $\int dt\, e^{(\omega_i - \omega_f)t}$, and hence we can write

$$2\pi[\delta(\omega_i - \omega_f)]^2 = \delta(\omega_i - \omega_f)\int_{-\infty}^{\infty} dt\, e^{i(\omega_i - \omega_f)t} = \delta(\omega_i - \omega_f)\int_{-\infty}^{\infty} dt$$

Defining a transition probability per unit time,[1] W_{fi}, by

$$\eta_{fi} \equiv W_{fi}\int_{-\infty}^{\infty} dt$$

we get the "golden rule"

$$W_{fi} = 2\pi\delta(\omega_i - \omega_f)\,|T_{fi}|^2 \tag{8.27}$$

The total transition rate out of the initial state can be written as

$$W_i = 2\pi \mathop{\mathrm{S}}\limits_{f} \delta(\omega_i - \omega_f)\,|T_{fi}|^2 = -2\,\mathrm{Im}\,T_{ii} \tag{8.28}$$

where the last equality is a consequence of (8.22),

$$\mathop{\mathrm{S}}\limits_{f} S^*_{fj}S_{fi} = \delta_{ij}$$

These formal expressions can be analyzed in more detail by diagonalizing the S matrix. In practice this is done by finding a sufficient number of constants of the motion whose eigenstates are also eigenstates of S. We define a projection operator $\mathfrak{P}^{(A)}$ onto eigenstates of S, denoted by A, by treating the energy separately, e.g.,

$$\delta_{fi} = \pi g(\omega_i)\delta(\omega_i - \omega_f)\mathop{\mathrm{S}}\limits_{A}\mathfrak{P}^{(A)}_{fi} \qquad g(\omega_i) = \frac{1}{\mathop{\mathrm{S}}\limits_{f}\pi\delta(\omega_i - \omega_f)\mathop{\mathrm{S}}\limits_{A}\mathfrak{P}^{(A)}_{fi}} \tag{8.29}$$

and write

$$S_{fi} = \pi g(\omega_i)\delta(\omega_i - \omega_f)\mathop{\mathrm{S}}\limits_{A} e^{2i\delta_A(\omega_i)}\mathfrak{P}^{(A)}_{fi} \tag{8.30}$$

$$T_{fi} = -g(\omega_i)\mathop{\mathrm{S}}\limits_{A} \sin\delta_A(\omega_i)e^{i\delta_A(\omega_i)}\mathfrak{P}^{(A)}_{fi}$$

Inserting into (8.28), we find, with $[\mathfrak{P}^{(A)}]^\dagger = \mathfrak{P}^{(A)}$,

$$W_i = 2g(\omega_i)\mathop{\mathrm{S}}\limits_{A}\mathfrak{P}^{(A)}_{fi}\sin^2\delta_A(\omega_i) \tag{8.31}$$

where $\delta(\omega_i)$ is the phase shift at the energy ω_i. This formalism simplifies greatly for a spherical source and only one outgoing particle per incoming one. Then S is diagonal in an angular-momentum representation, and ω is the only continuous variable:[2]

$$S\,|\,k, l, m\rangle = e^{2i\delta_l(k)}\,|\,k, l, m\rangle$$

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[1] A more satisfactory way of deducing these results is to consider wave packets. For a more complete discussion, see M. Gell-Mann and M. L. Goldberger, *Phys. Rev.*, **91**:70 (1953).

[2] The factor $\pi g(\omega_i)$ corresponds to the energy normalization

$$\langle k, l, m\,|\,k', l', m'\rangle = \pi g(\omega)\delta(\omega - \omega')\delta_{ll'}\delta_{mm'}$$

For further reference, see, e.g., B. A. Lippmann and J. Schwinger, *Phys. Rev.*, **79**:48 (1950).

or, in terms of the generating operators,

$$B_{klm} = S^{-1} A_{klm} S = e^{2i\delta_l(k)} A_{klm} \tag{8.32}$$

In the above case, we get

$$\mathsf{S}_{A} = \sum_{l=0}^{\infty}$$

$$\mathfrak{P}^{l}_{k'k} = \sum_{m=-l}^{l} Y_l^{m^*}(k') Y_l^{m}(k)$$

Furthermore, if we use a normalization volume L^3 and relativistic dynamics $k\,dk = \omega\,d\omega$, we have

$$\mathsf{S}_{f} = L^3 \int \frac{d^3k}{(2\pi)^3}$$

and

$$\frac{1}{g(\omega)} = L^3\pi \int \frac{d^3k'\ \delta(\omega' - \omega) Y_0^2}{(2\pi)^3} = \frac{L^3 \omega k}{8\pi^2}$$

Inserting into (8.31), we find, with

$$\sum_m |Y_l^m(0)|^2 = \frac{2l+1}{4\pi}$$

$$W_k = \frac{4\pi}{L^3 k\omega} \sum_l (2l+1) \sin^2 \delta_l(\omega) \tag{8.33}$$

To obtain the familiar expression for the cross section, we have to divide by the incident flux, that is, the number of incident particles per unit time and unit normal area ($= k/\omega L^3$):

$$\sigma_k = W_k \frac{\omega L^3}{k} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l(\omega) \tag{8.34}$$

In cases where particles are produced, there are other continuous variables, in addition to the energy, which characterize the eigenstates of S. In that case (8.34) has to be modified, but (8.27) and (8.28) are still valid.

Whereas the S matrix contains the information of the elementary phase-shift analysis, the matrix connecting the in states with the bare states corresponds to what is called the "wave matrix" in elementary wave mechanics. The latter contains information about the detailed form of the wave function in the near zone (i.e., near the source). In field theory it can answer questions about the distribution of virtual particles in physical states. Although such problems are largely of academic interest, they are instructive and will be studied in later chapters. Furthermore, we shall see that there are important relations between the S matrix and the wave matrix.

CHAPTER 9

Static Source

9.1. Interpretation of "Static" Source. As a first example, we shall carry out a detailed study for a static source $\rho(\mathbf{r},t) = g\rho(\mathbf{r})$. We assume this source to be centered about the origin of coordinates, to be real, and $\rho(\mathbf{r})$ to be normalized,

$$\int \rho(\mathbf{r})\, d^3r = 1$$

so that g represents the dimensionless strength of the source. In this case we encounter in the general solution (8.10) the expression

$$\int_{-\infty}^{\infty} dt\, \Delta^{\mathrm{ret}}(\mathbf{r},t) = \int_{-\infty}^{\infty} dt \mathop{\mathrm{S}}_{\mathbf{k}} \int \frac{dK_0}{2\pi} \frac{e^{i(\mathbf{k}\cdot\mathbf{r} - K_0 t)}}{m^2 + k^2 - K_0^2} \tag{9.1a}$$

If we can interchange the order of integration, then

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, \Delta^{\mathrm{ret}}(\mathbf{r},t) &= \mathop{\mathrm{S}}_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 + m^2} = \left(\frac{1}{2\pi}\right)^2 \int_0^{\infty} \frac{k^2\, dk}{k^2 + m^2} (e^{ikr} - e^{-ikr}) \frac{1}{ikr} \\ &= \left(\frac{1}{2\pi}\right)^2 \frac{1}{ir} \int_{-\infty}^{\infty} \frac{k\, dk\, e^{ikr}}{(k + im)(k - im)} = \frac{e^{-mr}}{4\pi r}\end{aligned} \tag{9.1b}$$

To the extent that this interchange of integration is permitted we shall get exactly the same result for $\int dt\, \Delta^{\mathrm{adv}}(\mathbf{r},t)$. These answers are equivalent to the statement that, for Δ^{adv} or Δ^{ret},

$$(\nabla^2 - m^2) \int_{-\infty}^{\infty} dt\, \Delta^{\mathrm{ret}}(\mathbf{r},t) = -\delta^3(\mathbf{r})$$

and $e^{-mr}/4\pi r$ is the Green's function for $(\nabla^2 - m^2)$.

The justification for the interchange of the K_0 and t integrations in (9.1a) rests on the meaning of a static source. The classes of interactions discussed in Chap. 8 demand that the source be zero at $t = \pm\infty$.

This is clearly not the case for a truly static source. It turns out, however, that switching the source on and off slowly (compared with m^{-1}) leads to the same results as a static source, and it is in this sense that (9.1b) is correct. To see this, we consider a source

$$\rho(\mathbf{r},t) = g\rho(\mathbf{r})e^{-\alpha|t|} \qquad \alpha > 0 \tag{9.2}$$

If we make use of (8.8), then the time integral which appears in (8.10) becomes

$$\int_{-\infty}^{\infty} \Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t')\,dt' = \mathop{\mathbf{S}}\limits_{\mathbf{k}} g\rho(\mathbf{r}')\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\omega}\int_{-\infty}^{t} e^{-\alpha|t'|}\sin\omega(t-t')\,dt'$$

$$= \mathop{\mathbf{S}}\limits_{\mathbf{k}} g\rho(\mathbf{r}')\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\omega}\begin{cases} \dfrac{\omega}{\alpha^2+\omega^2}e^{\alpha t} & \text{for } t<0 \\[2ex] \dfrac{\omega e^{-\alpha t} + 2\alpha\sin\omega t}{\alpha^2+\omega^2} & \text{for } t>0 \end{cases} \tag{9.3a}$$

In the limit of $\alpha \to 0$, or in fact $\alpha \ll m$, Eq. (9.3a) reduces to

$$\int_{-\infty}^{\infty} \Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t')\,dt' = g\rho(\mathbf{r}')\mathop{\mathbf{S}}\limits_{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\omega^2}$$

$$= g\rho(\mathbf{r}')\frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{9.3b}$$

which agrees with (9.1b). The same equation holds for Δ^{adv} in this limit. We thus note that a static source is to be interpreted as one which is switched on and off slowly during a time $\alpha^{-1} \gg m^{-1}$, and we then obtain, from (9.3b) or (9.1b),

$$\phi(\mathbf{r},t) = \phi^{\text{in}}(\mathbf{r},t) + g\int\Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}')\,dt'\,d^3r'$$

$$= \phi^{\text{in}}(\mathbf{r},t) + g\int d^3r'\,\frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')$$

$$= \phi^{\text{out}}(\mathbf{r},t) + g\int d^3r'\,\frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}') \tag{9.4}$$

We thus find that $\phi^{\text{in}} = \phi^{\text{out}}$, which means that the source creates no real particles and that there is no scattering. This is connected with the static form of the source and the lack of internal degrees of freedom.[1] That the process of switching on and off creates no disturbance (e.g., produces no particles) corresponds to the adiabatic theorem in elementary quantum theory, according to which a disturbance that varies slowly compared with the natural frequency of the system (which here is m) will not produce any transitions.

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[1] This conclusion will be clarified in subsequent chapters.

9.2. Energy of the Coupled System. To learn something about the eigenstates of the total energy, we shall first express the Hamiltonian H in terms of ϕ and use (8.18) to obtain

$$H = H_0 + H' = \frac{1}{2}\int d^3r\,[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 - 2g\rho(\mathbf{r})\phi]$$
$$= \sum_{\mathbf{k}} \left[a_{\mathbf{k}}^\dagger(t)a_{\mathbf{k}}(t)\omega - g\,\frac{\rho_{\mathbf{k}}^* a_{\mathbf{k}}(t) + \rho_{\mathbf{k}} a_{\mathbf{k}}^\dagger(t)}{(2\omega L^3)^{\frac{1}{2}}} \right] \tag{9.5}$$

In (9.5) we have denoted the Fourier transform[1] of $\rho(\mathbf{r})$ by $\rho_{\mathbf{k}}$,

$$\rho_{\mathbf{k}} = \int \rho(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3r \tag{9.6}$$

and its complex conjugate by $\rho_{\mathbf{k}}^*$. We can also express H in terms of the operators ϕ^{in}. If we make use of

$$(-m^2 + \nabla^2)\int \Delta^{\text{ret}}(\mathbf{r},t)\, dt = -\delta^3(\mathbf{r})$$

we find that

$$\int d^3r\, d^3r'\, [\nabla_r\phi^{\text{in}}(\mathbf{r},t)\nabla_r + m^2\phi^{\text{in}}]\,\frac{e^{-|\mathbf{r}-\mathbf{r}'|m}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}\,\rho(\mathbf{r}') = \int d^3r\, \phi^{\text{in}}(\mathbf{r},t)\rho(\mathbf{r})$$

Because of this equality, the cross terms that occur in H between ϕ^{in} and $\rho(\mathbf{r})$ vanish,[2] and we obtain

$$H = H^{\text{in}} + \mathscr{E}_0 \tag{9.7a}$$

with
$$H^{\text{in}} = \frac{1}{2}\int d^3r\,[(\dot{\phi}^{\text{in}})^2 + (\nabla\phi^{\text{in}})^2 + m^2(\phi^{\text{in}})^2]$$
$$= \sum_{\mathbf{k}} A_{\mathbf{k}}^\dagger A_{\mathbf{k}}\omega \tag{9.7b}$$

and
$$\mathscr{E}_0 = \frac{-g^2}{2}\int d^3r\, d^3r'\, \rho(\mathbf{r})\,\frac{e^{-m|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}\,\rho(\mathbf{r}')$$
$$= \frac{-g^2}{2}\, \mathop{\mathrm{S}}_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}|^2}{\omega^2} \tag{9.7c}$$

That $\mathscr{E}_0$ is a c number, commuting with ϕ^{in}, was to be anticipated. Since both ϕ^{in} and ϕ^{out} have the free-field time dependence, the Hamiltonian, when expressed in terms of either of these operators, must

---

[1] If $\rho(\mathbf{r})$ is normalized according to $\int \rho(\mathbf{r})\, d^3r = 1$, then $\rho_{\mathbf{k}}$ is dimensionless. Furthermore, $\rho_0 = 1$, and in the continuum limit $\rho(\mathbf{k}) = \rho_{\mathbf{k}}$. The Fourier transform of $\rho_{\mathbf{k}}$ is

$$\rho(\mathbf{r}) = \frac{1}{L^3}\sum_{\mathbf{k}} \rho_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = \left(\frac{1}{2\pi}\right)^3 \int \rho(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3k$$

[2] This can also be shown in momentum space. In the following, it is to be understood that the zero-point energy of the vacuum is subtracted from H.

reduce to $H^{(\text{in})}$ and a part which commutes with $\phi^{(\text{in})}$. We note that $H^{(\text{in})}$ and $\mathscr{E}_0$ are time-independent, but both H_0 and H' are not. In terms of the operators introduced by (8.17), we get (the zero-point energy $\Sigma\frac{1}{2}\omega$ has already been subtracted)

$$H = \sum_{\mathbf{k}} A_{\mathbf{k}}^{\dagger} A_{\mathbf{k}} \omega + \mathscr{E}_0 \tag{9.8}$$

The eigenvalue spectrum of H is, therefore, of the same form as that for the free fields, except that it is shifted down by an energy $|\mathscr{E}_0|$, as shown in Fig. 9.1. This energy represents a "binding" energy of the virtual particles, although the interaction energy H' is by no means a normal potential. That the physical ground-state energy $\mathscr{E}_0$ is less than zero shows that the problem considered belongs to the wide class of interactions which decrease the ground-state energy. This is always true when perturbation theory is applicable, since then[1]

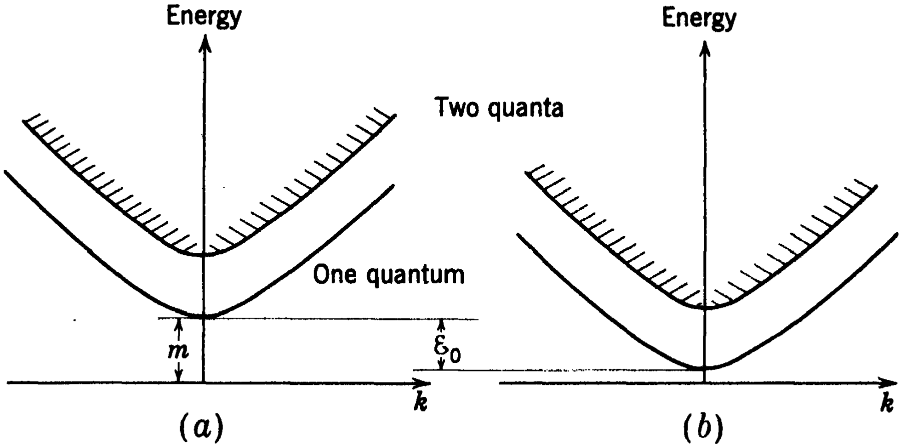

Fig. 9.1. The eigenvalue spectrum of H_0 is shown in part a and that of H in part b. One- and two-quantum states are represented. The energy shift between the two figures is $\mathscr{E}_0$.

$$\mathscr{E}_0 = -\sum_i \frac{|H'_{i0}|^2}{E_i - E_0} \tag{9.9}$$

Physically, this means that the ground state finds a way to take advantage of the new situation to lower its energy.

The energy $\mathscr{E}_0$ is referred to as an "energy renormalization" or sometimes as a "mass renormalization." The latter term is to be understood in the light of the following considerations, which use the equivalence of mass and energy in the sense of a relativistic theory. One can consider the static source to have a mechanical mass $M_0 \gg m$. The Hamiltonian H then becomes $H_0 + H' + M_0$. The bare eigenstates (of $H_0 + M_0$) then represent the source and the field without interaction. When the latter is turned on, then the lowest physical state is an eigenstate of H with energy $M_0 + \mathscr{E}_0 = M$. The mass M is the physical mass of the source and differs from the mechanical mass due to the interaction between the source and the field. It is in this sense, also, that eigenstates of H_0 are referred to as bare states and the eigenstates of H as physical states.

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[1] See L. I. Schiff, "Quantum Mechanics," 2d ed., p. 153, McGraw-Hill Book Company, Inc., New York, 1955. The matrix elements are between virtual or bare states, and it can be shown that, for the present problem, (9.9) gives the same answer as (9.7c).

For a point source, $\rho(\mathbf{r}) = \delta^3(\mathbf{r})$, we find that $\rho_\mathbf{k} = 1$ and that the energy $\mathscr{E}_0$ diverges linearly. There is, however, no way of observing this energy, no matter what its value, so long as the source is always surrounded by its cloud of quanta.[1] Since $\mathscr{E}_0$ is not observable, we can subtract it from H, so that

$$\mathscr{H} \,|\, \text{in}, 0\rangle \equiv (H - \mathscr{E}_0) \,|\, \text{in}, 0\rangle = 0 \tag{9.10}$$

9.3. Connection between Bare and Physical States. The (real, physical) ground state $|\,\text{in}, 0\rangle$ is again defined by

$$A_\mathbf{k} \,|\, \text{in}, 0\rangle = 0 \tag{9.11}$$

and the various eigenstates of the Hamiltonian are created by repeated applications of $A_\mathbf{k}^\dagger$. They correspond to a certain number of incoming particles with definite momenta. But $\phi^{\text{in}} = \phi^{\text{out}}$ or $A_\mathbf{k} = B_\mathbf{k}$, so that they also correspond to the same configuration of outgoing particles. Hence there is no scattering or creation of particles in this model.

Further insight into the model can be obtained by analyzing the incoming vacuum state $|\,\text{in}, 0\rangle$ in terms of eigenstates of $N(0)$. That is to say, we are interested in the configuration of virtual particles present in the physical ground state of the system at the time $t = 0$. To that purpose, we express $A_\mathbf{k}$ in terms of the $a_\mathbf{k}$ [defined by (8.18)]. This can be done by substituting (8.17) and (8.18) in (9.4) at $t = 0$ [we abbreviate $a_\mathbf{k}(0)$ as $a_\mathbf{k}$]:

$$\begin{aligned} \phi(\mathbf{r},0) \equiv \phi(\mathbf{r}) &= \sum_\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}} a_\mathbf{k} + e^{-i\mathbf{k}\cdot\mathbf{r}} a_\mathbf{k}^\dagger}{(2\omega L^3)^{\frac{1}{2}}} \\ &= \sum_\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}} A_\mathbf{k} + e^{-i\mathbf{k}\cdot\mathbf{r}} A_\mathbf{k}^\dagger}{(2\omega L^3)^{\frac{1}{2}}} + g \sum_\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}} \rho_\mathbf{k} + e^{-i\mathbf{k}\cdot\mathbf{r}} \rho_\mathbf{k}^*}{(2\omega^2 L^3)} \end{aligned}$$

from which we infer that

$$a_\mathbf{k} = A_\mathbf{k} + g \frac{\rho_\mathbf{k}}{(2\omega^3 L^3)^{\frac{1}{2}}} \tag{9.12}$$

and

$$a_\mathbf{k}^\dagger = A_\mathbf{k}^\dagger + g \frac{\rho_\mathbf{k}^*}{(2\omega^3 L^3)^{\frac{1}{2}}}$$

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[1] This will be made clear in the following discussion.

Therefore (9.11) tells us that

$$a_{\mathbf{k}}\,|\,\text{in}, 0\rangle = g\,\frac{\rho_{\mathbf{k}}}{(2\omega^3L^3)^{\frac{1}{2}}}\,|\,\text{in}, 0\rangle \equiv \chi_{\mathbf{k}}\,|\,\text{in}, 0\rangle \tag{9.13}$$

and this has the same form as the definition of our standard wave packet for the harmonic oscillator (2.23). The problem of analyzing $|\,\text{in}, 0\rangle$ in terms of the eigenstates $|\,n_{\mathbf{k}})$ is identical[1] with the calculation (2.25), and hence, with the notation of Sec. 2.3, we obtain

$$|\,(n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots, n_{\mathbf{k}_j}\,|\,\text{in}, 0\rangle|^2 = \prod_i \exp(-\bar{n}_{\mathbf{k}_i})\,\frac{(\bar{n}_{\mathbf{k}_i})^{n_{\mathbf{k}_i}}}{n_{\mathbf{k}_i}!} \tag{9.14}$$

with

$$\bar{n}_{\mathbf{k}} = g^2\,\frac{|\rho_{\mathbf{k}}|^2}{2\omega^3L^3}$$

This represents the probability for finding $n_{\mathbf{k}_1}$ virtual particles with momentum $\mathbf{k}_1$, $n_{\mathbf{k}_2}$ virtual particles with momentum $\mathbf{k}_2$, etc. The product form expresses the independence of the particles. The probability for finding $n_{\mathbf{k}_1}$ particles with momentum $\mathbf{k}_1$ irrespective of the number of particles with other momenta is given by the sum over all other $n_{\mathbf{k}_i}$:

$$\begin{aligned} \eta(n_{\mathbf{k}_1}) &= \sum_{n_{\mathbf{k}_2}\cdots} |(n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots, n_{\mathbf{k}_j}\,|\,\text{in}, 0\rangle|^2 \\ &= \exp(-\bar{n}_{\mathbf{k}_1})\,\frac{(\bar{n}_{\mathbf{k}_1})^{n_{\mathbf{k}_1}}}{n_{\mathbf{k}_1}!} \end{aligned} \tag{9.15}$$

That is to say, we have a Poisson distribution for the number of virtual particles of a definite momentum. Similarly, we derive by induction that the probability of finding n virtual particles irrespective of their momenta is

$$\eta(n) = \sum_{\sum_i n_{\mathbf{k}_i} = n} |\,(n_{\mathbf{k}_1}, \ldots, n_{\mathbf{k}_j}\,|\,\text{in}, 0\rangle\,|^2 = e^{-\bar{n}}\,\frac{(\bar{n})^n}{n!} \tag{9.16a}$$

with

$$\bar{n} = \sum_i \bar{n}_{\mathbf{k}_i} = g^2 \mathop{\mathrm{S}}\limits_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}|^2}{2\omega^3} \tag{9.16b}$$

which is again a Poisson law. The same law holds for the probability of finding n particles within a certain region Δ in momentum space, in which case $\bar{n}$ equals $\sum_{\Delta} \bar{n}_{\mathbf{k}_i}$. The number $\bar{n}$ represents the average number of field quanta which dress the source. As we shall see in the next chapter, it is this number which will be produced if the source is suddenly turned off. For a reasonably small source size (e.g., of radius $a \sim 1/k_{\max} \ll 1/m$), we find

$$\bar{n} \approx \frac{g^2}{4\pi}\,\frac{1}{\pi}\ln\frac{k_{\max}}{m} \sim \frac{g^2}{4\pi} \tag{9.16c}$$

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[1] The time dependence, of course, is now governed by H and not by H_0.

Thus the average number of virtual particles that surround the source is of the order of $g^2/4\pi$.

It is instructive to carry out the expansion of the physical ground state in a complete set of eigenfunctions of $N(t)$ at $t = 0$ and to compare this with the ground state of the hydrogen atom. By means of (9.13) and (9.16), we obtain

$$\begin{aligned} | \text{in}, 0\rangle &= \sum_n | n)(n | \text{in}, 0\rangle \\ &= | 0)(0 | \text{in}, 0\rangle + \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger | 0)(0 | a_{\mathbf{k}} | \text{in}, 0\rangle \\ &\quad + \sum_{\mathbf{k},\mathbf{k}'} \frac{a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger}{(2!)^{\frac{1}{2}}} | 0)(0 | \frac{a_{\mathbf{k}'} a_{\mathbf{k}}}{(2!)^{\frac{1}{2}}} | \text{in}, 0\rangle + \cdots \\ &= (0 | \text{in}, 0\rangle \left[| 0) + \sum_{\mathbf{k}} \chi_{\mathbf{k}} a_{\mathbf{k}}^\dagger | 0) + \sum_{\mathbf{k},\mathbf{k}'} \frac{1}{2!} \chi_{\mathbf{k}} \chi_{\mathbf{k}'} a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger | 0) + \cdots \right] \\ &= e^{-\bar{n}/2} \exp\left(\sum_{\mathbf{k}} \chi_{\mathbf{k}} a_{\mathbf{k}}^\dagger\right) | 0) \end{aligned} \tag{9.17}$$

where $\chi_{\mathbf{k}}$ is defined as in (9.13). The Fourier transform of the hydrogen-atom ground-state wave function is $\propto (1 + r_b^2 k^2)^{-2} \equiv \chi'_{\mathbf{k}}$, where r_b is the Bohr radius. In our notation this state could be written as

$$| \text{in}, 0\rangle = \sum_{\mathbf{k}} \chi'_{\mathbf{k}} a_{\mathbf{k}}^\dagger | 0) \tag{9.18}$$

In contrast to this, the ground state of the field is a mixture of states with various numbers (from 0 to ∞) of virtual particles. This fluctuation of the number of virtual particles is sometimes expressed by saying that the source creates and reabsorbs virtual particles. This terminology is similar to that used for the H_2^+ molecule, where we say that the electron is exchanged between the protons. The virtual particles are not always present, so that $e^{-\bar{n}/2}\chi_{\mathbf{k}}$, which corresponds to the wave function of a single virtual particle in momentum space, is not normalized to unity but to $\bar{n}e^{-\bar{n}} < 1$. The wave functions for states with several particles are simple products, which shows that the particles are uncorrelated except for effects due to the Bose-Einstein statistics. For a point source ($\rho_{\mathbf{k}} = 1$) the Fourier transform of the wave function $\chi_{\mathbf{k}} \sim \rho_{\mathbf{k}}/(k^2 + m^2)^{\frac{3}{4}}$ behaves approximately like $e^{-mr}/r^{\frac{3}{2}}$. More generally, we see that the expectation value for the field $\phi(\mathbf{r})$ in the ground state is just

$$\begin{aligned} \langle \text{in}, 0 | \phi(\mathbf{r}) | \text{in}, 0\rangle &= \mathop{\mathsf{S}}_{\mathbf{k}} \langle \text{in}, 0 | \frac{g\rho_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + g\rho_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r}}}{2\omega^2} | \text{in}, 0\rangle \\ &= \mathop{\mathsf{S}}_{\mathbf{k}} \int d^3r'\, g\rho(\mathbf{r}') \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{\omega^2} = g \int d^3r' \, \frac{e^{-|\mathbf{r}-\mathbf{r}'|m}}{4\pi |\mathbf{r} - \mathbf{r}'|} \quad \rho(r') \end{aligned} \tag{9.19}$$

so that the cloud of virtual field quanta covers the source with a thin veil of extension $\sim 1/m$, as shown in Fig. 9.2. (For pions, $m^{-1} \sim 10^{-13}$ cm.) This is required by the uncertainty principle, since virtual mesons cannot last longer than m^{-1} and can, therefore, not get farther than m^{-1}. Of course, their density is not sharply cut off after m^{-1} but decays exponentially. Such behavior is similar to the leakage of α particles into the energetically forbidden zone in α decay or the leakage of light into the dense medium in total reflection. Roughly speaking, we may say that all space outside the source is energetically forbidden for the virtual particles but that they can leak out because of the uncertainty relation.

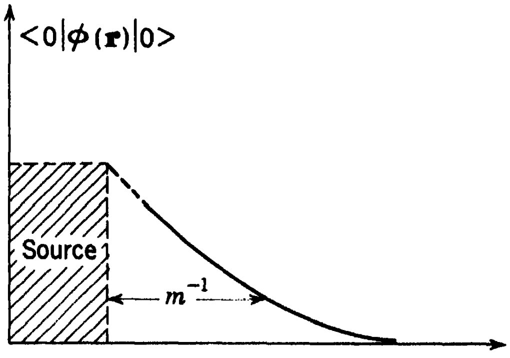

Fig. 9.2. Plot of $\langle 0 \mid \phi \mid 0\rangle$ close to the source.

It is the cloud of particles surrounding the source which dresses the latter and is responsible for the energy shift $\mathscr{E}_0$, which thus plays the role of a binding energy for these virtual particles. The meaning of the above becomes clear if we remember that, in the manipulations leading to (9.7), half of the contribution of the interaction energy H' [(see (9.5)] to $\mathscr{E}_0$ was canceled from the part stemming from H_0. Hence we have

$$\langle \text{in}, 0 \mid H_0 \mid \text{in}, 0\rangle = |\mathscr{E}_0| = -\tfrac{1}{2}\langle \text{in}, 0 \mid H' \mid \text{in}, 0\rangle \tag{9.20}$$

and this expresses a "virial theorem" if we call H' the potential energy. It states that the total energy which is kinetic (H_0) + potential energy is just the negative of the kinetic energy. Indeed, by means of (9.14) and (9.20) we find that the ground-state expectation value of H_0,

$$\langle \text{in}, 0 \mid H_0 \mid \text{in}, 0\rangle = \sum_{\mathbf{k}} \bar{n}_{\mathbf{k}}\omega = g^2 \mathop{S}_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}|^2}{2\omega^2} \tag{9.21}$$

is just the mean value of the kinetic energies (including the rest mass of the virtual particles). With this wider concept of a potential energy, we may say that virtual particles are bound with an energy which exceeds their mass m.

9.4. Fluctuations of the Field. For the square fluctuation of the field, our formula (4.13) is still valid in the present case. The reason for this is that the contribution from the source to $\langle 0 \mid \phi^2 \mid 0\rangle$ is canceled by $|\langle 0 \mid \phi \mid 0\rangle|^2$, which is not zero here. Here, as in the case of our standard wave packet (in Chap. 3), a mean number of mesons $\bar{n} \gg 1$ implies that the fluctuations of the field are less than its average value. The classical field picture can then be used. We have remarked before (see Chap. 4) that this is not the case for the electric field of an elementary point charge for which

$$\langle 0 | \phi | 0 \rangle \sim \frac{e}{4\pi r} < \frac{1}{r}$$

whereas the fluctuation of the field averaged over a region $\sim r^3$ is $\Delta\bar{\phi} \sim 1/r$; the mean number of photons in a volume Δ in momentum space, bounded by $k_{\max}$ and $k_{\min}$, is found from (9.16) to be

$$\bar{n}_\Delta = \frac{e^2}{4\pi^2} \int_{k_{\min}}^{k_{\max}} \frac{dk}{k} = \frac{e^2}{4\pi^2} \ln \frac{k_{\max}}{k_{\min}} \tag{9.22}$$

For reasonable $k_{\max}$ and $k_{\min}$ (e.g., $k_{\max} \sim m_{\mathrm{el}}$, $k_{\min} \sim 1/r_b$) we obtain, with $e^2/4\pi = \frac{1}{137}$, $\bar{n}_\Delta \sim \frac{1}{137} \ll 1$. However, charged macroscopic bodies always create the classical situation $\bar{n} \gg 1$. For the meson-nucleon system we shall see that $\bar{n} \sim 1$ and that 1 is in an awkward transition region.

It may happen that, for a particular form of the source, $\bar{n} \to \infty$, as it does for a point source. In this case the probabilities (9.14) for finding a finite number of virtual particles are zero. This implies that states with finite numbers of real particles are orthogonal to states with finite numbers of virtual particles. Thus, a perturbation or other expansion of a real state in terms of states of virtual particles is impossible; in particular, this applies to the ground-state expansion (9.17). If a perturbation expansion is nevertheless attempted, then infinities are always met. This difficulty is encountered in relativistic theories wherein the interactions must be localized. In a nonrelativistic theory, where the source may have a finite size, this problem may be circumvented.

9.5. Several Sources. Because (9.7*c*) is quadratic in ρ, the self-energies of several sources are not simply additive; there will be cross terms. In particular, for two spatially separated point sources of equal strength,

$$\rho(\mathbf{r}) = g[\delta^3(\mathbf{r}) + \delta^3(\mathbf{r} - \mathbf{r}_0)] \tag{9.23}$$

we find from (9.7) that

$$\mathscr{E}_0(2) = 2\mathscr{E}_0 - g^2 \frac{e^{-m|\mathbf{r}_0|}}{4\pi\,|\mathbf{r}_0|} \tag{9.24}$$

and from (9.17) that

$$| 0, 0 \rangle = e^{-\bar{n}/2} \exp \left[-\sum_{\mathbf{k}} \left(\frac{1}{2\omega^3 L^3} \right)^{\frac{1}{2}} \rho_{\mathbf{k}} g (1 + e^{i\mathbf{k}\cdot\mathbf{r}_0}) a_{\mathbf{k}}^\dagger \right] | 0, 0)$$

where $\mathscr{E}_0$ is the infinite, but constant, energy for one point source, given by $(-g^2/2) \sum_{\mathbf{k}} (1/\omega^2)$, and $| 0, 0 \rangle$ is the physical ground state of the two sources. The dependence of $\mathscr{E}_0$ (2) on the source separation $\mathbf{r}_0$ is of the form of the famous Yukawa potential.[1] In the language of perturbation theory, Eq. (9.9), it arises from the exchange of a virtual quantum between the two sources, and its range reflects the limitations imposed on such a process by the uncertainty relation. That is, an interaction between the two sources arises only when the distance between them is of the order of the size of the quantum cloud. One cannot, therefore, ascribe a classical path to the virtual particles, and the term "exchange" has to be taken with a grain of salt. It really implies that an overlap occurs for the virtual clouds belonging to the two sources. The limit of applicability of the classical concepts must be kept in mind in using intuitive pictures of the "exchange." The sign of the "potential" energy implies an attractive force between the two sources, since $\mathscr{E}_0(2) < 2\mathscr{E}_0$. This arises because the presence of a second source within a short distance of the first one opens channels for new processes that decrease the energy. However, this is true only if g has the same sign for both sources. If we were to take

$$\rho(\mathbf{r}) = g[\delta^3(\mathbf{r}) - \delta^3(\mathbf{r} - \mathbf{r}_0)] \tag{9.25}$$

the force would change sign. This obviously stems from the fact that, on close approach, the sources neutralize each other, which decreases $|\mathscr{E}_0(2)|$. We shall see in the next part that for the pion-nucleon system the situation is somewhat more complex and, depending on the spins and charges, the nucleon sources may have the same or the opposite "mesonic charge" g. Consequently the exchange of a meson will lead to an attractive "potential" in some states and to a repulsive one in others.

To conclude this section, we remark that for N point sources of arbitrary strength,

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} g_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{9.26}$$

we obtain

$$\mathscr{E}_0(N) = \sum_{i=1}^{N} \mathscr{E}_0^{(i)} - \sum_{i>j}^{N} \sum_{j=1}^{N} g_i g_j \frac{e^{-m|\mathbf{r}_i - \mathbf{r}_j|}}{4\pi |\mathbf{r}_i - \mathbf{r}_j|} \tag{9.27}$$

where $\mathscr{E}_0^{(i)} = -\dfrac{g_i^2}{2} \sum_{\mathbf{k}} \dfrac{1}{\omega^2}$

Thus the potential

$$\mathscr{E}_0(N) - \sum_{i=1}^{N} \mathscr{E}_0^{(i)}$$

is just the sum of the potentials between pairs, which shows that the presence of other sources does not disturb the force between a given pair. This is not generally true but is connected with the fact that quanta are not scattered in this model.

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[1] We shall postpone to the last chapter a discussion of the extent to which this term can actually be interpreted as the potential energy of the source particles.

Further Reading

The problem of one or more static sources can also be discussed in different but equivalent ways—e.g., by just stating a unitary transformation which diagonalizes the Hamiltonian. For this type of approach and for further studies the reader may refer to the following:

G. Wentzel, "Quantum Theory of Fields," p. 47, Interscience Publishers, Inc., New York, 1949.
S. Tomonaga, *Progr. Theoret. Phys.* (*Kyoto*), **2**:6 (1947).
L. Van Hove, *Physica*, **18**:145 (1952).
T. D. Lee, *Phys. Rev.*, **95**:1329 (1954).

CHAPTER 10

Production of Particles

10.1. General Remarks. Having studied the properties of virtual particles in the last chapter, we shall now investigate the circumstances in which they can be converted into real particles. To this end, we need to express outgoing operators in terms of incoming ones. According to the general formula (8.12), the connection between the incoming and outgoing field is given by

$$\phi^{\text{out}} = \phi^{\text{in}} + \int d^3r'\, dt'\, \Delta(\mathbf{r} - \mathbf{r}', t - t')\rho(\mathbf{r}',t') \tag{10.1}$$

where

$$\begin{aligned} \Delta(\mathbf{r},t) &= \Delta^{\text{ret}}(\mathbf{r},t) - \Delta^{\text{adv}}(\mathbf{r},t) \\ &= \mathop{\mathsf{S}}_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega} \sin \omega t \qquad \text{for } -\infty < t < \infty \end{aligned} \tag{10.2}$$

The relation (10.1) can be expressed in momentum space by making use of

$$\begin{aligned} \int \Delta(\mathbf{r} - \mathbf{r}', t - t')\rho(\mathbf{r}',t')\, d^3r'\, dt' &= \int \mathop{\mathsf{S}}_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \sin \omega(t - t')}{\omega}\, \rho(\mathbf{r}',t')\, d^3r'\, dt' \\ &= -\mathop{\mathsf{S}}_{\mathbf{k}} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{2i\omega}\, \rho_{\mathbf{k}}(\omega) + \mathop{\mathsf{S}}_{\mathbf{k}} \frac{e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{2i\omega}\, \rho_{\mathbf{k}}^{*}(\omega) \end{aligned}$$

where $\rho_{\mathbf{k}}$ is

$$\rho_{\mathbf{k}}(K_0) = \int \rho(\mathbf{r},t) e^{-i(\mathbf{k}\cdot\mathbf{r}-K_0 t)} d^3r\, dt \tag{10.3a}$$

It then reads

$$B_{\mathbf{k}} = S^{\dagger} A_{\mathbf{k}} S = A_{\mathbf{k}} + \frac{i\rho_{\mathbf{k}}(\omega)}{(2\omega L^3)^{\frac{1}{2}}} \qquad B_{\mathbf{k}}^{\dagger} = A_{\mathbf{k}}^{\dagger} - \frac{i\rho_{\mathbf{k}}^{*}(\omega)}{(2\omega L^3)^{\frac{1}{2}}} \tag{10.3b}$$

Since $B_{\mathbf{k}}$ obeys the free-field equation, only that part of the source contributes to it for which the frequency and wave number are related as for a free particle. Correspondingly, in ordinary space, the wave functions of the created particles will not be confined to a region close to the source. In classical terminology, we can say that the virtual particles are the ones contained in the near zone, whereas the outgoing particles leave the source and get into the wave zone of the field. The problem of analyzing states with a certain number of incoming particles in terms of the out states—in particular, the probability of finding $n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots$ outgoing particles with momenta $\mathbf{k}_1, \mathbf{k}_2, \ldots$ in the incoming vacuum—is again given by (9.14), since $B_{\mathbf{k}}\,|\text{in}, 0\rangle = i\rho_{\mathbf{k}}(\omega)(2\omega L^3)^{-\frac{1}{2}}|\text{in}, 0\rangle$, where $\bar{n}_{\mathbf{k}}$ now corresponds to

$$\bar{n}_{\mathbf{k}} = \frac{|\rho_{\mathbf{k}}(\omega)|^2}{2\omega L^3}$$

Hence we have a Poisson distribution of emitted quanta in every momentum interval Δ with a mean number of particles[1]

$$\bar{n}_\Delta = \mathop{\mathrm{S}}_{\mathbf{k}<\Delta} \frac{|\rho_{\mathbf{k}}(\omega)|^2}{2\omega} \tag{10.4}$$

We can also carry out an expansion exactly analogous to (9.17) and obtain

$$|\,\text{in}, 0\rangle = S\,|\,\text{out}, 0\rangle = e^{-\bar{n}/2} \exp\left[i \sum_{\mathbf{k}} \chi_{\mathbf{k}}(\omega) B_{\mathbf{k}}^{\dagger}\right] |\,\text{out}, 0\rangle \tag{10.4a}$$

with

$$\chi_{\mathbf{k}}(\omega) = \frac{\rho_{\mathbf{k}}(\omega)}{(2\omega L^3)^{\frac{1}{2}}} \tag{10.4b}$$

and

$$\bar{n} = \mathop{\mathrm{S}}_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}(\omega)|^2}{2\omega} \tag{10.4c}$$

We can also give an explicit expression for the S matrix in terms of the asymptotic field operators. In the case of one degree of freedom the S matrix corresponds to the unitary transformation $q \to q, p \to p + d$, which is generated by e^{iqd}. The field theoretic generalization of this is[2]

$$\begin{aligned} S &= e^{-\bar{n}/2} \exp\left[i \sum_{\mathbf{k}} \chi_{\mathbf{k}}(\omega) B_{\mathbf{k}}^{\dagger}\right] \exp\left[i \sum_{\mathbf{k}} \chi_{\mathbf{k}}^{*}(\omega) B_{\mathbf{k}}\right] \\ &= \exp\left\{i \sum_{\mathbf{k}} [\chi_{\mathbf{k}}(\omega) B_{\mathbf{k}}^{\dagger} + \chi_{\mathbf{k}}^{*}(\omega) B_{\mathbf{k}}]\right\} \end{aligned} \tag{10.4d}$$

---

[1] $\rho_{\mathbf{k}}(\omega) = \rho(\mathbf{k},\omega)$.

[2] Care must be exercised in writing exponential operators. Since $B_{\mathbf{k}}$ and $B_{\mathbf{k}}^{\dagger}$ do not commute, $e^{B_k + B_k^{\dagger}} \neq e^{B_k} e^{B_k^{\dagger}}$, as can be seen by expanding the exponentials. However, because the commutator of $B_{\mathbf{k}}$ and $B_{\mathbf{k}}^{\dagger}$ is a c number, we have

$$\begin{aligned} \exp\left[\sum_{\mathbf{k}} (B_{\mathbf{k}}^{\dagger} f_{\mathbf{k}} + B_{\mathbf{k}} g_{\mathbf{k}})\right] &= \exp\left(\sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger} f_{\mathbf{k}}\right) \exp\left(\sum_{\mathbf{k}} B_{\mathbf{k}} g_{\mathbf{k}}\right) \exp\left(\sum_{\mathbf{k},\mathbf{k}'} [B_{\mathbf{k}}, B_{\mathbf{k}'}^{\dagger}] g_{\mathbf{k}} f_{\mathbf{k}'}/2\right) \\ &= \exp\left(\sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger} f_{\mathbf{k}}\right) \exp\left(\sum_{\mathbf{k}} B_{\mathbf{k}} g_{\mathbf{k}}\right) \exp\left(\sum_{\mathbf{k}} g_{\mathbf{k}} f_{\mathbf{k}}/2\right) \end{aligned}$$

By means of (10.3*b*) the matrix S can be expressed in terms of the operators of the incoming field $A_{\mathbf{k}}$ and $A_{\mathbf{k}}^{\dagger}$ as

$$S = \exp\left\{i \sum_{\mathbf{k}} [\chi_{\mathbf{k}}(\omega) A_{\mathbf{k}}^{\dagger} + \chi_{\mathbf{k}}^{*}(\omega) A_{\mathbf{k}}]\right\} \tag{10.4e}$$

The reader can convince himself that this expression is consistent with (10.4*a*) and that

$$B_{\mathbf{k}} = S^{\dagger} A_{\mathbf{k}} S \qquad B_{\mathbf{k}}^{\dagger} = S^{\dagger} A_{\mathbf{k}}^{\dagger} S$$

If the source $\rho(\mathbf{r},t)$ is spherically symmetric, then the only contribution to the S matrix arises from spherically symmetric or angular-momentum zero terms. That is, if an expansion of $B_{\mathbf{k}}$ is made in terms of the angular-momentum operators B_{klm}, then only B_{k00} contributes to the S matrix. Hence all particles are produced in S states (angular momentum zero).

Before discussing this expression for typical forms of ρ, we shall answer another question that might arise at this point. We know that for a time-dependent source the energy of the field is not conserved. Then what is the energy pumped into (or taken out of) the field by the source? For a vacuum of incoming particles this quantity is the expectation value of the energy in this state as $t \to \infty$. It can be evaluated in the following manner:

$$\begin{aligned}
&\langle \text{in}, 0 \,|\, H^{\text{out}} | \text{in}, 0 \rangle \\
&\quad = \langle \text{in}, 0 \,|\, \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger} B_{\mathbf{k}} \omega \,|\, \text{in}, 0 \rangle \\
&\quad = \sum_{n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \ldots} \; \sum_{n'_{\mathbf{k}_1}, n'_{\mathbf{k}_2}, \ldots} \langle \text{in}, 0 \,|\, \text{out}, n_{\mathbf{k}_i} \rangle \langle \text{out}, n_{\mathbf{k}_i} | H^{\text{out}} | \text{out}, n'_{\mathbf{k}_j} \rangle \langle \text{out}, n'_{\mathbf{k}_j} \,|\, \text{in}, 0 \rangle \\
&\quad = \sum_{n_{\mathbf{k}_i}} \sum_{\mathbf{k}} n_{\mathbf{k}_i} \omega \, |\langle \text{in}, 0 \,|\, \text{out}, n_{\mathbf{k}_i} \rangle|^2
\end{aligned}$$

Since $|\langle \text{in}, 0 \,|\, \text{out}, n_{\mathbf{k}_i} \rangle|^2$ is a Poisson distribution, use of (6.15) gives

$$\langle \text{in}, 0 \,|\, H^{\text{out}} \,|\, \text{in}, 0 \rangle = \sum_{\mathbf{k}} \omega \bar{n}_{\mathbf{k}} = \mathop{\mathsf{S}}_{\mathbf{k}} \tfrac{1}{2} |\rho_{\mathbf{k}}(\omega)|^2 \tag{10.5}$$

That is, the energy transferred to the field is just equal to the total energy of the quanta created by the source. In classical field theory we obtain (10.5) by integrating $\partial H / \partial t = -\partial L / \partial t$ from $t = -\infty$ to $t = +\infty$. Since $\rho(\mathbf{r},t)$ is the only term in the Lagrangian which depends explicitly on the time, we obtain

$$\Delta E = -\int_{-\infty}^{\infty} dt \int d^3r \; \phi(\mathbf{r},t) \frac{\partial}{\partial t} \rho(\mathbf{r},t)$$

Classically, to an incoming vacuum there corresponds an incoming field $\phi^{\text{in}} = 0$ (quantum-mechanically, its expectation value is zero). By a partial integration, substitution of (8.10), and use of (8.9), we can rewrite ΔE as

$$\begin{aligned}\Delta E &= \int_{-\infty}^{\infty} dt \int d^3r\, d^3r'\, dt'\, \rho(\mathbf{r},t)\rho(\mathbf{r}',t')\frac{\partial}{\partial t}\Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}',t-t') \\ &= \int dt\, dt'\, d^3r\, d^3r'\, \rho(\mathbf{r},t)\rho(\mathbf{r}',t')\frac{\partial}{\partial t}\frac{1}{2}[\Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}',t-t') - \Delta^{\text{adv}}(\mathbf{r}-\mathbf{r}',t-t')]\end{aligned} \tag{10.6}$$

By going over to momentum space and using (10.2), we verify (10.5). In the last way of writing (10.6), we have emphasized that it is the difference between the incoming and outgoing fields which is relevant for the energy loss. This is well known from classical electrodynamics, where a similar expression for the energy loss obtains and it turns out that the radiative force is generated by[1] $\phi^{\text{rad}} = \phi^{\text{out}} - \phi^{\text{in}}$. Since the expectation values of the field obey the classical equation of motion and the fluctuation terms are the same as for the free fields, it was to be expected that we should obtain the classical result for the energy loss. Similarly, for a static source the self-energy $\mathscr{E}_0$ is found to be the energy change obtained by switching the source on or off.

10.2. Specific Examples. It is clear from (10.4) that particles are created only if the source contains frequencies[2] $>m$. For instance, for a point source with a periodic time dependence,

$$\rho(\mathbf{r},t) = g\delta^3(\mathbf{r})\cos\omega_0 t \tag{10.7a}$$

and hence

$$\rho_{\mathbf{k}}(K_0) = g\pi[\delta(\omega_0 - K_0) + \delta(\omega_0 + K_0)] \tag{10.8a}$$

$\bar{n}$, as given by (10.4*c*), is zero unless $\omega_0 > m$. This is again a consequence of the adiabatic principle according to which a quasistatic source should be almost as good as a static one. For (10.8*a*) $\bar{n}$ becomes $\propto |\delta(\omega_0 - \omega)|^2$ and therefore infinite. This is physically clear, since such a source keeps radiating particles from $t = -\infty$ to $t = +\infty$. To obtain a finite result, let the source radiate only during a finite period,

$$\rho(\mathbf{r},t) = g\delta^3(\mathbf{r})\, e^{-\alpha|t|}\cos\omega_0 t \tag{10.7b}$$

$$\rho_{\mathbf{k}}(K_0) = g\left[\frac{\alpha}{\alpha^2 + (K_0 - \omega_0)^2} + \frac{\alpha}{\alpha^2 + (K_0 + \omega_0)^2}\right] \tag{10.8b}$$

in which case the number of particles is proportional to the time during which the source radiates. For $\alpha \ll m$ the four-dimensional Fourier transform of the source, $\rho_{\mathbf{k}}(\omega)$, reduces to (10.8*a*), and we can write

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[1] See, e.g., W. Thirring, "Principles of Quantum Electrodynamics," chap. 2, Academic Press, Inc., New York, 1958.

[2] Thus, the static source (which is switched on and off adiabatically) has no such frequencies.

$$\bar{n} \approx \tfrac{1}{2} \mathop{S}_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}(\omega)|^2}{\omega} = \frac{g^2\pi}{\alpha} \int \frac{d^3k}{(2\pi)^3} \frac{\delta(\omega_0 - \omega)}{2\omega}$$

$$= \frac{g^2}{4\pi} \frac{(\omega_0^2 - m^2)^{\frac{1}{2}}}{\alpha} \qquad \text{if } \omega_0 > m \tag{10.9a}$$

This result can also be deduced by calculating the outgoing current of particles.

The physical significance of ϕ^{out} is illustrated by calculating its expectation value for the incoming vacuum. For a spherical source it consists of spherical waves. This becomes obvious when the expansion (5.10) is used in eigenstates of the angular momentum, since such a source couples only to that part of the field which has angular momentum zero.[1] Furthermore, for times after the source has been switched off, ϕ^{out} contains only outgoing waves, as we should expect intuitively. These statements are most easily verified for a source of the form (10.7*b*) and $m = 0$. Then, $\Delta(\mathbf{r},t)$ becomes $\sim r^{-1}[\delta(r - t) - \delta(r + t)]$, and we find

$$\langle \text{in}, 0|\phi^{\text{out}}(\mathbf{r},t)|\text{in}, 0\rangle = \frac{g}{4\pi r}\left[\cos{(r - t)\omega_0}\, e^{-\alpha|r-t|} - \cos{(r + t)\omega_0} e^{-\alpha|r+t|}\right]$$

For large positive t, only the first term with an outgoing spherical wave persists.

Returning to our example with $m \neq 0$, we find that for small values of α the number of particles created per unit time is essentially $\bar{n}\alpha$, and the number created in a time interval δt is

$$\bar{n}\alpha\, \delta t = \frac{g^2}{4\pi} (\omega_0^2 - m^2)^{\frac{1}{2}}\, \delta t \qquad \text{if } \omega_0 > m \tag{10.9b}$$

This is analogous to the expression for radiation in classical electrodynamics and has a simple physical interpretation. We saw in Chap. 9 [see Eq. (9.16*c*)] that the number of virtual particles surrounding the source is of the order of $g^2/4\pi$. Since the wave number k is the velocity v multiplied by the energy ω, the right-hand side of Eq. (10.9*b*) can be written as

$$(g^2/4\pi)(v\, \delta t)(\omega)$$

or (number of virtual particles) × (distance particles with velocity v can go in time δt) × (radius of cloud of particles)$^{-1}$, and therefore represents that number of virtual particles which can leave with a velocity v within a time interval δt. This implies that as soon as the necessary energy is supplied, the source quanta start leaving the cloud with their final (real-particle) velocity. In this intuitive picture of the creation process, we must keep in mind that the stock of virtual particles in the field is inexhaustible, because they are automatically regenerated in the source. The time for regeneration is of the order ω_0^{-1}, and to get significantly more particles than are present in the cloud, we have to wait longer than this time.

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[1] This type of expansion will be carried out in subsequent chapters.

In a sudden event we can never get more than all the particles contained in the cloud, this number being obtained by suddenly switching off the source. Indeed, for this case,

$$\rho(\mathbf{r},t) = \begin{cases} g\rho(\mathbf{r}) & \text{for } t < 0 \\ 0 & \text{for } t > 0 \end{cases} \tag{10.10}$$

gives[1]

$$\rho_{\mathbf{k}}(K_0) = \frac{g\rho_{\mathbf{k}}}{iK_0}$$

and therefore

$$\bar{n} = g^2 \mathop{\mathrm{S}}_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}|^2}{2\omega^3} \tag{10.11}$$

which is just the number of virtual particles (9.16*b*) of the source $\rho(\mathbf{r})$. We explained in Chap. 9 that this result is quite general and can be used to define the number of virtual particles.

The scarcity of virtual photons in the field of an elementary charge is the reason why electric processes are so slow. A particle with unit charge and velocity $v \approx 1$ has to make of the order of 137 violent collisions until it shakes off a photon, since this quantum is present in the cloud less than 1 per cent of the time. Therefore the cross section for *Bremsstrahlung* is 137 times less than the collision cross section. We shall see that we shall get a simple understanding of pion-nucleon phenomena by picturing the nucleon dissociated, for a certain fraction of time, into a pion and a nucleon. This fraction is quite sizable, and so is the probability of having more than one meson around. Hence, almost every time the energy is available, a meson is emitted by the nucleon, and multiple production is also a fairly frequent event.

Another interesting idealization is a source which suddenly changes its velocity:[2]

$$\rho(\mathbf{r},t) = \begin{cases} \rho(\mathbf{r}) & \text{for } t < 0 \\ \rho(\mathbf{r} - \mathbf{v}t) & \text{for } t > 0 \end{cases} \tag{10.12}$$

$$\rho_{\mathbf{k}}(K_0) = -i\rho_{\mathbf{k}}\left(\frac{1}{K_0} - \frac{1}{K_0 - \mathbf{v}\cdot\mathbf{k}}\right) \tag{10.13}$$

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[1] The adiabatic switching-on process is assumed for $t < 0$; that is,

$$\rho(\mathbf{r},t) = \lim_{\alpha\to 0} g\rho(\mathbf{r})\, e^{\alpha t} \qquad \text{for } t < 0$$

[2] Equation (10.12) is not relativistically correct, since the Lorentz contraction of the source is neglected. The change in velocity may occur because of a collision of very short duration (e.g., two nucleons colliding at very high energy). In any case, v should be <1, since otherwise even the uniformly moving source radiates (Cerenkov radiation).

and hence

$$\bar{n} = \mathop{\mathrm{S}}_{\mathbf{k}} \frac{|\rho_{\mathbf{k}}|^2}{2\omega}\left(\frac{1}{\omega} - \frac{1}{\omega - \mathbf{v}\cdot\mathbf{k}}\right)^2 \tag{10.14}$$

The two terms in $\rho_{\mathbf{k}}(K_0)$ correspond to the source at rest and to the moving source. Hence it is just the number of virtual particles in the difference of the fields before and after $t = 0$ which accounts for the real particles at $t = \infty$. This agrees with our earlier remark that what is radiated is the difference between the incoming and the outgoing field.

For $m = 0$ and for a spherically symmetric source, (10.14) becomes ($\omega = k$, and θ is the angle between $\mathbf{v}$ and $\mathbf{k}$)

$$\bar{n} = \int_0^\infty \frac{dk\,|\rho_{\mathbf{k}}|^2}{(2\pi)^3\,2k}\,d\Omega\left(1 - \frac{1}{1 - v\cos\theta}\right)^2 \tag{10.15}$$

which exhibits the well-known *Bremsstrahlung*'s spectrum $\propto dk/k$ as long as $\rho_{\mathbf{k}} \sim 1$. For a point source the energy loss $\bar{n}_k k$ is proportional to $\int dk$, so that in every frequency interval the same amount of energy is radiated. In this case the expression for $\bar{n}$ diverges at both ends. The upper limit is easily fixed up by taking an extended source which averages out the very high frequencies. The integral diverges at the lower limit even for an extended source, since normalizing the source $\int d^3r\,\rho(\mathbf{r}) = 1$ implies $\rho_{k=0} = 1$. This divergence means that we always deal with infinitely many quanta (both real and virtual) of low frequency. For the virtual particles this can be understood as follows. For a point source the self-energy (in r space) is given by $\frac{1}{2}\int \mathfrak{E}^2\,d^3r$, where $\mathfrak{E}$ is the electric field strength, and this is $(e^2/4\pi)\int d^3r/8\pi r^4$. It diverges linearly at the lower end but converges at the upper end, in agreement with our expression (9.2), since the upper limit in k space corresponds to the lower limit in r space, and vice versa. The particle density has an extra power of k in the denominator or of r in the numerator and diverges at both limits as in (9.22). The infinity of infrared quanta corresponds to the $1/r$ behavior of the Coulomb field at large distance and is not removed by smearing out the source. Consequently, an infinity of quanta is radiated when the asymptotic parts of the Coulomb field are changed. The number of infrared quanta within the radius of the universe[1] is <1, and the "problem" is somewhat academic. It does not arise for $m \neq 0$, since then $\phi \sim e^{-mr}/r$ and has a finite range.

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[1] From (9.22),

$$\bar{n} \sim e^2 \ln\frac{k_{\max}}{k_{\min}} \sim e^2 \ln\frac{r_{\max}}{r_{\min}}$$

Taking $r_{\min}$ as the electromagnetic radius of the electron ($\sim 10^{-13}$ cm), we find

$$\bar{n} = 1 \qquad \text{for } r_{\max} = 10^{-13}e^{137} \sim 10^{+47}\ \text{cm}$$

However, $r_{\text{universe}} \sim 10^{27}$ cm.

CHAPTER 11

Pair Theory, Classical

11.1. General Remarks. In this chapter we shall turn our attention to a system wherein the interaction term is quadratic in the field variables. The general form of such a term, as discussed in Chap. 8, is

$$L'(t) = \frac{1}{2}\int d^3r\, d^3r'\, \phi(\mathbf{r},t)V(\mathbf{r},\mathbf{r}',t)\phi(\mathbf{r}',t) \tag{11.1}$$

which includes, as a special case, a local potential $V(\mathbf{r},\mathbf{r}',t) \propto \delta^3(\mathbf{r}' - \mathbf{r})$ acting on the field. However, there are cases where nonlocal potentials of the form (11.1) are important, as in the many-body problem of nuclear physics.[1] Furthermore, some types of linear couplings may effectively correspond to a coupling of the type (11.1). For instance, an electromagnetic field $\mathbf{A}$ interacting with nonrelativistic particles of charge e has a Lagrangian

$$L = \frac{1}{2m_0}\int d^3r\, \psi^*(\mathbf{r})[i\nabla - e\mathbf{A}(\mathbf{r})]^2\psi(\mathbf{r}) \tag{11.2}$$

which contains a term of the form (11.1) with

$$V(\mathbf{r},\mathbf{r}',t) = \frac{e^2}{m_0}\delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r},t) \tag{11.3}$$

$$\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$$

In a relativistic electromagnetic theory the situation is more complex;

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[1] See R. J. Eden, in P. M. Endt and M. Demeur (eds.), "Nuclear Reactions," vol. I, chap. 1, North-Holland Publishing Co., Amsterdam, 1959.

in particular, the δ function becomes smeared out over a Compton wavelength of the charged particles. Similarly, the relativistic pseudoscalar γ_5 interaction[1] in meson theory can be shown[2] to be equivalent to a leading term of the form (11.1) with

$$V(\mathbf{r},\mathbf{r}',t) = \frac{g^2}{M_0}\,\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r},t) \tag{11.4}$$

where ρ is the nucleon density and $\psi^*(\mathbf{r})\psi(\mathbf{r})$ and M_0 is the mass of the nucleon. Many other terms appear in the transformation, one of which will be the subject of the last part of this book. Also, that part of the Lee model which can be solved explicitly is equivalent to (11.1), as we shall see in Chap. 13. In this and the following chapter we shall frequently refer to electrodynamics as an example of such a theory. However, we shall study this problem not so much because its physical applications are of interest but because it conveniently introduces many concepts which we shall encounter in pion physics. Thus, the model allows us to investigate bound (excited) physical source states and scattering phenomena.

Following our general approach, we shall first study the solutions of the equations of motion from the point of view of the classical field and shall defer the application of quantum mechanics to the next chapter. We shall limit our discussion to the case for which V does not depend on t.¶ Furthermore, in any explicit evaluation we shall, for simplicity, always take V to be separable and of the form[3]

$$V(\mathbf{r},\mathbf{r}') = -\lambda\rho(\mathbf{r})\rho(\mathbf{r}') \tag{11.5}$$

$$\int d^3r\ \rho(\mathbf{r}) = 1$$

where λ is real and $\rho(\mathbf{r})$ is spherically symmetric. The development we shall present can also be carried out with the general form of V, but this requires more tools from the theory of singular integral equations. Since the general case does not lead to essential new features of physical interest, it will not be analyzed in detail. The equation of motion

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[1] The reader who has never heard of this is advised to ignore the following remark. The Lagrangian is $L' = \int \psi^*\beta\gamma_5\phi\psi\, d^3r$.

[2] F. J. Dyson, *Phys. Rev.*, **73**:929 (1948); L. L. Foldy, *Phys. Rev.*, **84**:168 (1951); S. D. Drell and E. M. Henley, *Phys. Rev.*, **88**:1053 (1952).

¶ Of course, we keep the liberty of switching it on and off adiabatically.

[3] In the equation the coupling constant λ has the dimension of length or (energy)$^{-1}$. We could equally well introduce a dimensionless coupling constant $g = \lambda m$. When λ is negative, the Hamiltonian is no longer positive definite, and difficulties may arise, as we shall see in the next chapter.

(8.12) is written in integral form by introducing the asymptotic fields:

$$\begin{aligned}\phi(\mathbf{r},t) &= \phi^{\text{in}}(\mathbf{r},t) + \int dt'\, d^3r'\, d^3r''\, \Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')V(\mathbf{r}',\mathbf{r}'')\phi(\mathbf{r}'',t') \\ &= \phi^{\text{out}}(\mathbf{r},t) + \int dt'\, d^3r'\, d^3r''\, \Delta^{\text{adv}}(\mathbf{r}-\mathbf{r}', t-t')V(\mathbf{r}',\mathbf{r}'')\phi(\mathbf{r}'',t') \qquad (11.6)\end{aligned}$$

The Fourier transform of this equation has the usual form of a linear integral equation. To obtain it, we introduce the four-dimensional Fourier transform[1] of the field ϕ:

$$\phi(\mathbf{k},K_0) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \phi(\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}-K_0t)}\, d^3r\, dt$$

and the three-dimensional Fourier transform of the time-independent source:[2]

$$V(\mathbf{k},\mathbf{k}') = \int V(\mathbf{r},\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}\, d^3r\, d^3r'$$

The reason that the four-dimensional Fourier transform, rather than the three-dimensional one, is introduced for the field ϕ is that its time dependence is not that of a free field and is not known a priori. Writing in terms of the above transforms and introducing four-dimensional Fourier transforms of Δ^{ret} or Δ^{adv}, as given by (8.6) and (8.7), we obtain[3]

$$\begin{aligned}\phi(\mathbf{k},K_0) &= \phi^{\text{in}}(\mathbf{k},K_0) + \mathop{\mathbf{S}}_{\mathbf{k}'} \frac{V(\mathbf{k},\mathbf{k}')\phi(\mathbf{k}',K_0)}{\omega^2-(K_0+i\epsilon)^2} \\ &= \phi^{\text{out}}(\mathbf{k},K_0) + \mathop{\mathbf{S}}_{\mathbf{k}'} \frac{V(\mathbf{k},\mathbf{k}')\phi(\mathbf{k}',K_0)}{\omega^2-(K_0-i\epsilon)^2} \qquad (11.7)\end{aligned}$$

Such an equation can be considered to be the limit of an ordinary linear

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[1] We shall henceforth use the continuum ($L \to \infty$) normalization but shall retain $\mathop{\mathbf{S}}_{\mathbf{k}} \to (1/2\pi)^3\, d^3k$ as a symbolic notation to indicate integration over the continuous spectrum as well as summation over the discrete spectrum, when it exists. The correspondence between the continuum and finite box normalization was made clear in Chap. 5. The normalization of $1/(2\pi)^{\frac{3}{2}}$ rather than $1/(2\pi)^2$ is made for the sake of convenience. The inverse transform is then

$$\phi(\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^{\frac{5}{2}} \int \phi(\mathbf{k},K_0)e^{i(\mathbf{k}\cdot\mathbf{r}-K_0t)}\, d^3k\, dK_0$$

[2] The normalization is again chosen so that for a separable source, as defined by (11.5), $V(0,0) = -\lambda$. The sign convention is also chosen for convenience and has the added feature that for a local potential $V(\mathbf{r},\mathbf{r}') \propto \delta^3(\mathbf{r}-\mathbf{r}')$, we find that $V(\mathbf{k},\mathbf{k}')$ depends only on the momentum difference $\mathbf{k}-\mathbf{k}'$.

[3] We have introduced $i\epsilon$ as a convenient shorthand notation to indicate the paths of integration discussed in Chap. 8. The limit $\epsilon \to 0$ is to be understood; hence $Ni\epsilon$, where N is any finite number larger than zero, can also be written as just $i\epsilon$.

algebraic equation (which it actually would be, had we used a finite normalization volume[1]) and can be treated by similar methods. For the separable source, (11.7) contains λ and K_0 as parameters and represents, in general, a linear relation between ϕ and ϕ^{in}. However, if the determinant of (11.7) vanishes for certain values of the parameters, then there is a solution with $\phi^{\text{in}} = 0$ or $\phi^{\text{out}} = 0$. We shall see later on that the solutions of the homogenous equations (e.g., $\phi^{\text{in}} = 0$) correspond to bound states, whereas the solutions of the inhomogenous equations are scattering states. For the moment we concentrate on the latter. In general, the solution of (11.7) can be given only as an infinite series, but for V of the form (11.5) the Fredholm series terminates after the second term. Indeed, we find that[2]

$$\phi(\mathbf{k},K_0) = \phi^{\text{in}}(\mathbf{k},K_0) + \lambda \mathop{\mathrm{S}}_{\mathbf{k}'} \frac{\rho(\mathbf{k})\rho^*(\mathbf{k}')}{(K_0 + i\epsilon)^2 - \omega^2}\,\phi(\mathbf{k}',K_0) \tag{11.8a}$$

$$\rho(\mathbf{k}) = \int \rho(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3r$$

$$\begin{aligned} \rho(-\mathbf{k}) &= \rho^*(\mathbf{k}) \qquad \text{for a real source} \\ \rho(\mathbf{k}) &= \rho(|\mathbf{k}|) \qquad \text{for a spherical source} \end{aligned} \tag{11.8b}$$

This can be integrated immediately to give

$$\phi(\mathbf{k},K_0) = \phi^{\text{in}}(\mathbf{k},K_0) + \lambda \mathop{\mathrm{S}}_{\mathbf{k}'} \frac{\rho(\mathbf{k})\rho^*(\mathbf{k}')}{(K_0 + i\epsilon)^2 - \omega^2}\,\frac{\phi^{\text{in}}(\mathbf{k}',K_0)}{1 + \lambda \mathop{\mathrm{S}}\limits_{\mathbf{q}} \dfrac{|\rho(\mathbf{q})|^2}{\omega_q^2 - (K_0 + i\epsilon)^2}} \tag{11.9}$$

Now that we have obtained a specific solution, we can return to the more familiar three-dimensional Fourier decomposition. To rewrite (11.9) in three-dimensional momentum space, we shall make use of our knowledge of the free-field time dependence of ϕ^{in}. In terms of the continuum analogue of the Fourier decomposition given by (8.16), we have

$$\begin{aligned} \phi^{\text{in}}(\mathbf{r},t) &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[\frac{A(\mathbf{k})e^{-i\omega t}}{(2\omega)^{\frac{1}{2}}} + \frac{A^\dagger(-\mathbf{k})e^{i\omega t}}{(2\omega)^{\frac{1}{2}}}\right] e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[\phi^{\text{in}(+)}(\mathbf{k},t) + \phi^{\text{in}(-)}(\mathbf{k},t)\right] e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \tag{11.9a}$$

where $\phi^{\text{in}(+)}(\mathbf{k},t)$ is the positive-frequency part[3] of ϕ^{in}, proportional to

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[1] For this case the problem is treated in G. Wentzel, *Helv. Phys. Acta*, **15**:111 (1942).

[2] We shall carry out the manipulations only for ϕ^{in}; the ones for ϕ^{out} follow the same pattern.

[3] This notation was introduced in Chap. 5.

$e^{-i\omega t}$, and $\phi^{\mathrm{in}(-)}(\mathbf{k},t)$ is the negative-frequency term, proportional to $e^{i\omega t}$. The four-dimensional Fourier transform of $\phi^{\mathrm{in}}(\mathbf{r},t)$, as defined earlier in this chapter, differs from zero only for $K_0 = \pm\omega$ and is

$$\phi^{\mathrm{in}}(\mathbf{k},K_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi^{\mathrm{in}}(\mathbf{k},t)e^{iK_0t}\,dt = \frac{A(\mathbf{k})}{(2\omega)^{\frac{1}{2}}}\,\delta(K_0-\omega) + \frac{A^{\dagger}(-\mathbf{k})}{(2\omega)^{\frac{1}{2}}}\,\delta(K_0+\omega)$$

Writing in terms of the positive- and negative-frequency components of ϕ^{in} and making use of the above results, we can rewrite (11.9) in a convenient matrix notation[1] as

$$\phi(\mathbf{k},t) = (\mathbf{k}\,|\,\Omega_+\,|\,\mathbf{k}')\phi^{\mathrm{in}(+)}(\mathbf{k}',t) + (\mathbf{k}\,|\,\Omega_-\,|\,\mathbf{k}')\phi^{\mathrm{in}(-)}(\mathbf{k}',t) \qquad (11.10)$$

with
$$(\mathbf{k}\,|\,\Omega_\pm\,|\,\mathbf{k}') = (2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}') + \frac{\lambda\rho(\mathbf{k})\rho^*(\mathbf{k}')}{(k'^2-k^2\pm i\epsilon)D_\pm(k'^2)} \qquad (11.11)$$

It follows from (11.8*b*) and (11.11) that for a real spherically symmetric source

$$\Omega_\pm = \Omega_\mp^*$$

$$D_\pm(k'^2) = 1 + \frac{\lambda}{(2\pi)^3}\int\frac{|\rho(\mathbf{q})|^2\,d^3q}{q^2-k'^2\mp i\epsilon} \qquad (11.12a)$$

$$D_\pm^*(k^2) = D_\mp(k^2) \qquad (11.12b)$$

We shall sometimes write (11.10) in the shorthand notation

$$\phi = \Omega\phi^{\mathrm{in}} \qquad (11.10a)$$

where the field equation (8.11) requires[2]

$$\bar\omega^2\Omega - \Omega\bar\omega^2 = V\Omega \qquad (11.13)$$

which can easily be checked. Here $\bar\omega$ is considered a diagonal matrix

$$(\mathbf{k}\,|\,\bar\omega\,|\,\mathbf{k}') = (2\pi)^{-3}\,\delta^3(\mathbf{k}-\mathbf{k}')\omega$$

and matrix multiplication with $V(\mathbf{k},\mathbf{k}')$ is implied on the right-hand side.

11.2. Bound States. To find the totality of the solutions of (11.7), we still have to consider the homogeneous solution. The function $D_\pm(k^2)$ defined by (11.12) is the Fredholm determinant of (11.7). As stated earlier, if it vanishes for certain eigenvalues of $k = k_b$, then a solution of (11.7) exists with $\phi^{\text{in}} = 0$. We shall subsequently study when this occurs, but for the moment we simply note that such a solution is[1]

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[1] Double subscripts are integrated over in the sense of $\underset{\mathbf{k}}{\mathsf{S}}$. The matrices occurring here are not to be confused with the operators in the Hilbert space of quantum mechanics.

[2] To differentiate between the matrix ω and the function ω, we place a bar over the former.

$$\phi(k,K_0) = \text{constant}\,\frac{\rho(k)}{k^2 - k_b^2}\,\delta(K_0 \pm \omega_b)$$

or

$$\phi(k,t) = \text{constant}\,\frac{\rho(k)}{k^2 - k_b^2}\,e^{\pm i\omega t} \tag{11.14}$$

This satisfies (11.7) provided that

$$\lambda \mathop{\mathrm{S}}\limits_{\mathbf{q}} \frac{|\rho(q)|^2}{k_b^2 - q^2} = 1$$

[e.g., $D(k_b^2) = 0$]. We shall shortly show that this solution represents a bound state, in that the classical field has the proper spatial behavior for such a state.

In order to find when (11.7) has a solution with $\phi^{\text{in}} = 0$, we have to study the properties of the function $D_{\pm}(k^2)$. For real values of k we may use[2]

$$\frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x) \tag{11.15}$$

to split $D_{\pm}$ into its real and imaginary parts, D_1 and D, respectively,

$$\begin{aligned} D_{\pm}(k^2) &= 1 + \lambda P\int \frac{d^3q\,|\rho(q)|^2}{(2\pi)^3(q^2 - k^2)} \pm i\lambda\frac{|\rho(k)|^2\,|k|}{4\pi} \\ &= D_1 \pm iD \end{aligned} \tag{11.16}$$

If $\rho(k)$ is a function that goes to zero smoothly[3] as $k \to \infty$, then (11.16) shows that D can never vanish for real values of k. To study the analytic behavior of D in the complex plane, we replace k^2 by the complex variable[4] $z = x + iy$. Then $D_{+}(k^2)$ and $D_{-}(k^2)$ are the boundary values of

$$D(z) = 1 + \lambda \mathop{\mathrm{S}}\limits_{\mathbf{q}} \frac{|\rho(q)|^2}{q^2 - z} \tag{11.17}$$

when the real axis is approached from above and below, respectively.

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[1] We shall assume that $\rho(\mathbf{k})$ is spherically symmetric throughout the remainder of this chapter and most of the succeeding one. Then $\rho(\mathbf{k}) = \rho(|\mathbf{k}|) = \rho(k)$.

[2] P means Cauchy's principal part.

[3] By this we mean that $\rho(k)$ is always positive. This would not be true for a patial source, $\rho(r)$, which cuts off sharply beyond a certain radial distance $r = r_0$.

[4] $\pm i\epsilon$ is then included in y.

From (11.17) it appears that $D(z)$ is analytic in the whole complex plane, save for a branch line along the positive real axis where the imaginary part is discontinuous. To find the zeros of $D(z)$, we note that

$$\mathrm{Im}\, D(z) = y\lambda \mathop{S}_{q} \frac{|\rho(q)|^2}{(q^2 - x)^2 + y^2} \tag{11.18}$$

is zero only for $y = 0$. To locate the zeros of $D(z)$, we therefore need to look only at the negative real axis. There we have[1]

$$\frac{1}{\lambda} D'(x) \equiv \frac{1}{\lambda} \frac{d}{dx} D(x) = \mathop{S}_{q} \frac{|\rho(q)|^2}{(q^2 - x)^2} > 0 \tag{11.19}$$

and since $D(\pm\infty) = 1$ if[2] $\int d^3q\, |\rho(q)|^2 < \infty$, we have in these circumstances:

1. $\lambda > 0$: D has no zeros in the cut plane.
2. $\lambda < 0$: D has one and only one zero for a negative real value of $x = k_b^2$ if

$$D(0) = 1 + \lambda \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(k)|^2}{k^2} < 0$$

otherwise none.

These conditions are shown graphically in Fig. 11.1. It was to be expected that bound states occur only for $\lambda < 0$, since this corresponds to an attractive interaction, and we see that our form of the potential admits only one bound state. It is well known from ordinary quantum mechanics that even a short-range attractive interaction will not have a bound state if the potential energy is too weak to overcome the kinetic zero = point energy. This is also true here for $\lambda < 0$ but $D(0) > 0$.

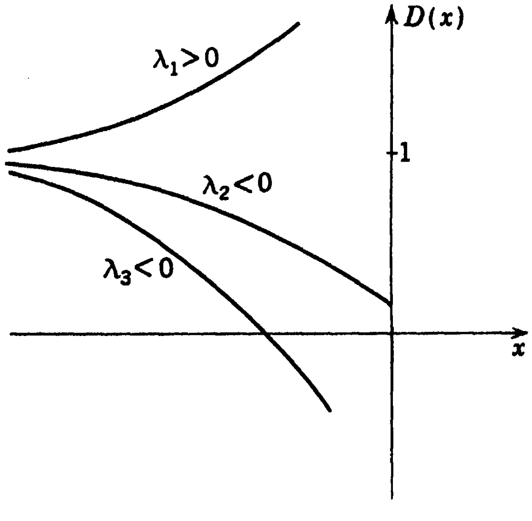

Fig. 11.1. Diagram for $D(x)$ for negative values of x and three possible values of λ, with $\lambda_1 > \lambda_2 > \lambda_3$ and λ_2 and λ_3 both < 0.

11.3. Behavior of the Wave Matrix Ω. The matrix[3] Ω which transforms the local field variables into asymptotic fields has some interesting properties which are essential for the quantum theoretic treatment.

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[1] We hope that the reader will not confuse $x = \mathrm{Re}\,(z)$, $y = \mathrm{Im}\,(z)$, and $\mathbf{r} \equiv (x,y,z)$.

[2] This condition is not satisfied for a point source.

[3] In the literature, Ω is sometimes called the wave matrix; see C. Møller, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.*, **23**(1) (1945).

We shall see that it is one-sided unitary if bound states[1] exist in the sense that[2]

$$\Omega_\pm^\dagger \Omega_\pm = 1 \tag{11.20}$$

but

$$\Omega_\pm \Omega_\pm^\dagger = 1 - \mathfrak{P} \tag{11.21}$$

where $\mathfrak{P}$ is the projection operator onto the bound states if any exist. The one-sided unitarity is a phenomenon which occurs only in an infinite-dimensional space, since for finite matrices $\Omega\Omega^\dagger = 1$ always implies $\Omega^\dagger\Omega = 1$. To prove these propositions, we write

$$\Omega_\pm = 1 + R_\pm \tag{11.22}$$

where

$$(\mathbf{k}' \mid R_\pm \mid \mathbf{k}) = \frac{\lambda}{D_\pm(k^2)} \frac{\rho(k')\rho^*(k)}{k^2 - k'^2 \pm i\epsilon}$$

and

$$(\mathbf{k}' \mid R_\pm^\dagger \mid \mathbf{k}) = \frac{\lambda}{D_\mp(k'^2)} \frac{\rho(k')\rho^*(k)}{k'^2 - k^2 \mp i\epsilon}$$

so that (11.20) becomes

$$R_\pm^\dagger R_\pm = -(R_\pm + R_\pm^\dagger) \tag{11.23}$$

This can be explicitly demonstrated for a separable potential as follows:

$$\begin{aligned}
&(\mathbf{k}' \mid R_+^\dagger R_+ \mid \mathbf{k}) \\
&\quad = \int \frac{d^3q}{(2\pi)^3} \frac{\lambda^2 \rho(k')\rho^*(k)}{D_-(k'^2)D_+(k^2)} \frac{|\rho(q)|^2}{(k'^2 - q^2 - i\epsilon)(k^2 - q^2 + i\epsilon)} \\
&\quad = \frac{\lambda\rho(k')\rho^*(k)}{(k^2 - k'^2 + i\epsilon)D_-(k'^2)D_+(k^2)} \int \frac{d^3q}{(2\pi)^3} \left[\frac{\lambda\,|\rho(q)|^2}{k'^2 - q^2 - i\epsilon} - \frac{\lambda\,|\rho(q)|^2}{k^2 - q^2 + i\epsilon}\right] \\
&\quad = \frac{\lambda\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon} \left[\frac{1}{D_-(k'^2)} - \frac{1}{D_+(k^2)}\right] \\
&\quad = -(\mathbf{k}' \mid R_+ + R_+^\dagger \mid \mathbf{k})
\end{aligned} \tag{11.24}$$

The same relations can be proved to hold for R_-.

The verification of (11.21) proceeds the same way, and we find

$$\begin{aligned}
&(\mathbf{k}' \mid R_+ R_+^\dagger \mid \mathbf{k}) \\
&\quad = \lambda^2 \int \frac{d^3q}{(2\pi)^3} \frac{\rho(k')\rho^*(k)\;|\rho(q)|^2}{(q^2 - k'^2 + i\epsilon)(q^2 - k^2 - i\epsilon)D_+(q^2)D_-(q^2)} \\
&\quad = \frac{\lambda\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon} \int \frac{d^3q}{(2\pi)^3} \frac{|\rho(q)|^2}{D_+(q^2)D_-(q^2)} \left(\frac{\lambda}{k'^2 - q^2 - i\epsilon} - \frac{\lambda}{k^2 - q^2 + i\epsilon}\right)
\end{aligned} \tag{11.25}$$

---

[1] The bound states have binding energies $< m$. The ground state of the physical source is excluded from these considerations. It always exists, unless the source is unstable, and we shall not consider such problems.

[2] Because of the matrix multiplication introduced earlier (see footnote 1, page 104), $\langle \mathbf{k} \mid \Omega_\pm^\dagger \Omega_\pm \mid \mathbf{k}' \rangle$ is not dimensionless. It is, in fact, equal to $(2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}')$, and the 1 in (11.20) and (11.21) is to be interpreted in this sense.

With (11.16) this last integral can be written

$$(\mathbf{k}' \,|\, R_+R_+^\dagger \,|\, \mathbf{k})$$

$$= \frac{\lambda\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon}\frac{1}{i\pi}\int_0^\infty q\,dq\left[\frac{1}{D_-(q^2)} - \frac{1}{D_+(q^2)}\right]\left(\frac{1}{k'^2 - q^2 - i\epsilon} - \frac{1}{k^2 - q^2 + i\epsilon}\right)$$

$$= \frac{\lambda\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon}\frac{1}{2\pi i}\int_C \frac{dz}{D(z)}\left(\frac{1}{z - k'^2 + i\epsilon} - \frac{1}{z - k^2 - i\epsilon}\right) \cdot \qquad (11.26)$$

where C is a contour going below the real axis from $+\infty$ to 0 and above it from 0 to ∞. Since $D(\infty) \to 1$, an integral over an infinite circle contributes nothing to (11.26) and can be added to the closed path shown in Fig. 11.2, to complete the contour. We can evaluate this by means of Cauchy's theorem, using the analytic properties of D discussed above. If D has no pole, we obtain

$$(\mathbf{k}' \,|\, R_+R_+^\dagger \,|\, \mathbf{k}) = \frac{\lambda\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon}\left[\frac{1}{D_-(k'^2)} - \frac{1}{D_+(k^2)}\right]$$

and if there is one, we find[1]

$$(\mathbf{k}' \,|\, R_+R_+^\dagger \,|\, \mathbf{k})$$

$$= \frac{\lambda\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon}\left[\frac{1}{D_-(k'^2)} - \frac{1}{D_+(k^2)} + \frac{1}{D'(k_b^2)}\left(\frac{1}{k_b^2 - k'^2} - \frac{1}{k_b^2 - k^2}\right)\right]$$

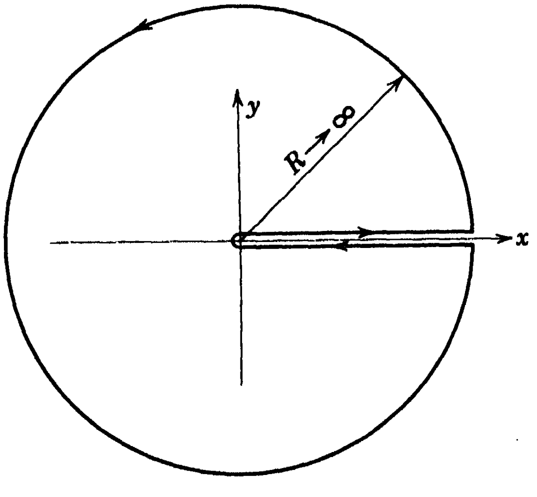

Fig. 11.2. Integration contour for Eq. (11.26).

Returning to (11.22), we see that the first case implies

$$(\mathbf{k}' \,|\, \Omega_+\Omega_+^\dagger \,|\, \mathbf{k}) = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \qquad (11.27a)$$

whereas the second one results in

$$(\mathbf{k}' \,|\, \Omega_+\Omega_+^\dagger \,|\, \mathbf{k}) = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') - U_b(k')U_b^*(k) \qquad (11.27b)$$

with[2]

$$U_b(k) = \left[\frac{\lambda}{D'(k_b^2)}\right]^{\frac{1}{2}}\frac{\rho(k)}{k^2 - k_b^2} \qquad (11.28a)$$

Comparison with (11.14) allows us to write the time dependence of U_b as

$$U_b(k,t) = U_b(k)e^{-i\omega t} \qquad (11.28b)$$

---

[1] $D'(k_b^2) = dD(x)/dx\,|_{x=k_b^2}$ and is given by (11.19).

[2] Note that λ must be negative for the second term to appear in (11.28). However, $D'(k_b^2)$ is also negative, as shown by (11.19), so that $\lambda/D'(k_b^2)$ is always positive.

It is a simple matter to demonstrate, with the aid of (11.19), that the wave function U_b of the bound state is correctly normalized. That this wave function does indeed represent a bound state follows from the spatial dependence of the Fourier transform of $U_b(k)$. Since k_b^2 is negative, we find

$$U_b(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int U_b(k) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3k = \left[\frac{\lambda}{D'(k_b^2)}\right]^{\frac{1}{2}} \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \rho(r')\, \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2 + |k_b^2|}\, d^3k\, d^3r'$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[\frac{\lambda}{D'(k_b^2)}\right]^{\frac{1}{2}} \int \rho(r')\, \frac{e^{-|k_b|\,|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \tag{11.28c}$$

and at large distances from the source this wave function has the characteristic spatial dependence of a bound state,

$$\lim_{r\to\infty} U_b(\mathbf{r}') = \left[\frac{\pi\lambda}{2\,D'(k_b^2)}\right]^{\frac{1}{2}} \frac{e^{-|k_b|r}}{r} \tag{11.28d}$$

where $\omega_b = (m^2 - |k_b^2|)^{\frac{1}{2}}$ is the binding energy. Furthermore, the wave function U_b is orthogonal to the scattering states[1] in the sense that

$$\int \frac{d^3k}{(2\pi)^3}\, U_b^*(k)(\mathbf{k} \mid \Omega_\pm \mid \mathbf{k}') = 0 \tag{11.29}$$

11.4. Scattering. To conclude this chapter, we turn to the connection between the incoming and the outgoing fields. From (11.10) and its analogue for outgoing fields, we obtain[2]

$$\begin{aligned}\phi(\mathbf{k},t) &= (\mathbf{k} \mid \Omega_+ \mid \mathbf{k}')\phi^{(+)\mathrm{in}}(\mathbf{k}',t) + (\mathbf{k} \mid \Omega_- \mid \mathbf{k}')\phi^{(-)\mathrm{in}}(\mathbf{k}',t)\\ &= (\mathbf{k} \mid \Omega_- \mid \mathbf{k}')\phi^{(+)\mathrm{out}}(\mathbf{k}',t) + (\mathbf{k} \mid \Omega_+ \mid \mathbf{k}')\phi^{(-)\mathrm{out}}(\mathbf{k}',t)\end{aligned} \tag{11.30}$$

Since this equation holds for all times (ϕ^{in} and ϕ^{out} have the free-field time dependence), we can split it into positive- and negative-frequency parts to obtain, by means of (11.20),

$$\begin{aligned}\phi^{(+)\mathrm{out}} &= \Omega_-^\dagger \Omega_+ \phi^{(+)\mathrm{in}}\\ \phi^{(-)\mathrm{out}} &= \Omega_+^\dagger \Omega_- \phi^{(-)\mathrm{in}}\end{aligned} \tag{11.31}$$

Because $[\phi^{(+)}]^\dagger = \phi^{(-)}$, the second form of (11.31) is consistent with the first one. It is important to recognize that $\Omega_-^\dagger\Omega_+$ is unitary whether there are bound states or not. We find, by means of (11.21),

$$\begin{aligned}\Omega_-^\dagger\Omega_+(\Omega_-^\dagger\Omega_+)^\dagger &= \Omega_-^\dagger(1 - \mathfrak{P})\Omega_- = 1\\ (\Omega_-^\dagger\Omega_+)^\dagger\Omega_-^\dagger\Omega_+ &= \Omega_+^\dagger(1 - \mathfrak{P})\Omega_+ = 1\end{aligned} \tag{11.32}$$

---

[1] That $(\mathbf{k} \mid \Omega_\pm \mid \mathbf{k}')$ represent scattering states will be made clear in Sec. 11.4. The proof of the orthogonality (11.29) is left as an exercise for the reader.

[2] The bound state is now ignored, since its presence would not change the following considerations.

since the term with $\mathfrak{P}$ drops out because of the orthogonality relation (11.29). If the source is spherically symmetric, $\rho(\mathbf{k}) = \rho(|\mathbf{k}|)$, as has been assumed, then $\Omega_-^\dagger \Omega_+$ can easily be diagonalized. As for the linear coupling, only the spherically symmetric part of the field is then coupled. This is exhibited by expanding the field in spherical waves, in which case Ω becomes diagonal in l and m. It is essentially unity for matrix elements with $|\, k, l, m)$ that have $l \neq 0$, and for $l = 0$ we find[1]

$$(k', 0, 0 \,|\, \Omega_+ \,|\, k, 0, 0) = \frac{1}{4\pi} \int d\Omega_k \, d\Omega_{k'} \, Y_0^0(k')(k' \,|\, \Omega_+ \,|\, k) Y_0^0(k)$$

$$= \frac{2\pi^2}{k^2} \delta(|k| - |k'|) + \frac{\lambda}{D_+(k^2)} \frac{\rho(k')\rho^*(k)}{k^2 - k'^2 + i\epsilon} \tag{11.33}$$

By means of (11.15) and (11.16) it is possible to cast (11.33) into a more convenient form:

$$(k', 0, 0 \,|\, \Omega_+ \,|\, k, 0, 0)$$
$$= \left[1 - \frac{i\lambda\,|k|}{4\pi} \frac{|\rho(k)|^2}{D_+(k^2)}\right] \delta(k - k') \frac{2\pi^2}{k^2} + \lambda P \frac{\rho(k')\rho^*(k)}{(k^2 - k'^2) D_+(k^2)}$$
$$= \left[\frac{2\pi^2}{k^2} \delta(k - k') + \lambda P \frac{\rho(k')\rho^*(k)}{(k^2 - k'^2) D_1(k^2)}\right] \frac{D_1(k^2)}{D_+(k^2)}$$
$$\equiv (k', 0, 0 \,|\, \Omega_1 \,|\, k, 0, 0) \frac{D_1(k^2)}{D_+(k^2)} \tag{11.34}$$

Similarly, we have

$$(k', 0, 0 \,|\, \Omega_- \,|\, k, 0, 0) = (k', 0, 0 \,|\, \Omega_1 \,|\, k, 0, 0) \frac{D_1(k^2)}{D_-(k^2)}$$
$$= (k', 0, 0 \,|\, \Omega_+ \,|\, k, 0, 0) \frac{D_+(k^2)}{D_-(k^2)} \tag{11.35}$$

and hence[2]

$$\Omega_-^\dagger \Omega_+ = \left(\Omega_+ \frac{D_+}{D_-}\right)^\dagger \Omega_+ = \frac{D_-}{D_+} \Omega_+^\dagger \Omega_+ = \frac{D_-}{D_+} 1$$

or

$$(k', 0, 0 \,|\, \Omega_-^\dagger \Omega_+ \,|\, k, 0, 0) = \frac{2\pi^2}{k^2} \delta(k - k') \frac{D_-(k^2)}{D_+(k^2)} \tag{11.36}$$

Since this last matrix is diagonal, the connection between the in and out fields is particularly simple in this representation. If we define the phase shift $\delta(k)$ by[1]

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[1] See Chap. 5.

[2] Matrix multiplication now implies $(1/2\pi^2)\int k^2 \, dk$.

$$(k', 0, 0 \,|\, \Omega_-^\dagger \Omega_+ \,|\, k, 0, 0) = \frac{2\pi^2}{k^2}\,\delta(k - k')e^{2i\delta(k)} \tag{11.37}$$

then
$$\phi^{(+)\mathrm{out}} = e^{2i\delta(k)}\phi^{(+)\mathrm{in}} \qquad \phi^{(-)\mathrm{out}} = e^{-2i\delta(k)}\phi^{(-)\mathrm{in}} \tag{11.38}$$

and
$$\tan\delta(k) = -\frac{D(k^2)}{D_1(k^2)} = -\frac{\lambda}{4\pi}\,\frac{|\rho(k)|^2\,k}{1 + \lambda P\displaystyle\int \frac{d^3q}{(2\pi)^3}\frac{|\rho(q)|^2}{q^2 - k^2}} \tag{11.39}$$

These equations will be of importance in the next chapter, where we shall study the particle aspect of the problem and shall relate these expressions to scattering cross sections. Here we simply wish to remark that difficulties are encountered when the source size shrinks to zero. Since $\rho(k)$ is then equal to 1, we find, from (11.39),

$$\lim_{k\to 0}\frac{\tan\delta(k)}{k} = -\frac{\lambda}{4\pi}\frac{1}{\infty} = 0 \tag{11.40}$$

It will be shown in Chap. 12 how these difficulties can be circumvented for observables by a limiting procedure.

As for the linear coupling, pair theory can also be solved analytically for several sources. This is a rewarding problem, since it allows one to study the scattering on many centers. Furthermore, the "potential energy" is not just the sum of potentials between pairs, as in the linear theory. It would, however, carry us too far afield to discuss these problems here.[2]

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[1] That this is the usual phase shift (8.32) follows from (11.38). It is in the above sense that Ω represents the scattering "wave function." It is normalized according to (11.20) and (11.21).

[2] See, however, Wentzel, *loc. cit.*

CHAPTER 12

Pair Theory, Quantum-mechanical

12.1. Quantization and Commutation Relations in the Presence of a Bound State. We shall first use our previous results to check the consistency of the local field commutation relations with those for the asymptotic field. Although, in general, this equivalence is assured by the adiabatic principle, it is not so obvious for the pair theory as for the linear coupling, where the local and asymptotic field operators differ only by an ordinary number. In particular, if we immediately consider the more general problem, which includes a bound state, then the canonical commutation relations require that, in addition to the incoming field (11.10), ϕ contain a term which represents the bound state. As we saw in Chap. 11 [see (11.28*c*)], the spatial dependence of the bound-state wave function is

$$U_b(\mathbf{r},t) \propto \int d^3r'\, \rho(\mathbf{r}')\, \frac{e^{-|k_b|\,|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, e^{-i\omega_b t}$$

with $$\omega_b = (m^2 + k_b^2)^{\frac{1}{2}} = (m^2 - |k_b|^2)^{\frac{1}{2}}$$

so that the corresponding particles remain concentrated around the source in the limit $t \rightarrow \pm\infty$, whereas those associated with ϕ^{in} eventually disappear at infinity. An *Ansatz* which, as we shall see, satisfies the commutation relations is (always employing the matrix notation introduced in the last section)

$$\phi(\mathbf{k},t) = (\mathbf{k}\,|\,\Omega_+\,|\,\mathbf{k}')\phi^{(+)\text{in}}(\mathbf{k}',t) + U_b(\mathbf{k})\phi_b(t) + (\mathbf{k}\,|\,\Omega_-\,|\,\mathbf{k}')\phi^{(-)\text{in}}(\mathbf{k}',t) + U_b^*(\mathbf{k})\phi_b^\dagger(t) \qquad (12.1)$$

where ϕ^{in} is the usual operator (8.17) for the incoming field and

$$\begin{aligned} \phi_b(t) &= \frac{A_b e^{-i\omega_b t}}{(2\omega_b)^{\frac{1}{2}}} \\ [A_b, A_{\mathbf{k}}] &= [A_b, A_{\mathbf{k}}^\dagger] = 0 \qquad (12.2) \\ [A_b, A_b^\dagger] &= 1 \end{aligned}$$

With these commutation relations and the familiar ones for ϕ^{in}, we can convince ourselves that the canonical commutation rules are satisfied. This would not be true if the terms proportional to $U_b(\mathbf{k})$ were omitted or treated as c numbers. For simplicity, we shall carry out the proof for $t = 0$; it will be evident that things work the same way for other times. From (12.1) and (11.9a) we deduce

$$\begin{aligned} \phi(\mathbf{k},0) &= (2\pi)^{-\frac{3}{2}}(\mathbf{k} \,|\, \Omega_+(2\bar{\omega})^{-\frac{1}{2}} \,|\, \mathbf{k}')A(\mathbf{k}') + U_b(\mathbf{k})A_b(2\omega_b)^{-\frac{1}{2}} \\ &\quad + (2\pi)^{-\frac{3}{2}}(\mathbf{k} \,|\, \Omega_-(2\bar{\omega})^{-\frac{1}{2}} \,|\, \mathbf{k}')A^\dagger(-\mathbf{k}') + U_b^*(\mathbf{k})A_b^\dagger(2\omega_b)^{-\frac{1}{2}} \\ i\dot{\phi}(\mathbf{k},0) &= (2\pi)^{-\frac{3}{2}}\left(\mathbf{k} \,\middle|\, \Omega_+\left(\frac{\bar{\omega}}{2}\right)^{\frac{1}{2}} \,\middle|\, \mathbf{k}'\right)A(\mathbf{k}') + U_b(\mathbf{k})A_b\left(\frac{\omega_b}{2}\right)^{\frac{1}{2}} \qquad (12.3) \\ &\quad -(2\pi)^{-\frac{3}{2}}\left(\mathbf{k} \,\middle|\, \Omega_-\left(\frac{\bar{\omega}}{2}\right)^{\frac{1}{2}} \,\middle|\, \mathbf{k}'\right)A^\dagger(-\mathbf{k}') - U_b^*(\mathbf{k})A_b^\dagger\left(\frac{\omega_b}{2}\right)^{\frac{1}{2}} \end{aligned}$$

In terms of the Fourier transform $\phi(\mathbf{k},0)$ and $\dot{\phi}(\mathbf{k},0)$ introduced in Chap. 11, the reality of $\phi(\mathbf{r})$ requires that $\phi(-\mathbf{k},0) = \phi^\dagger(\mathbf{k},0)$, and the usual commutation relations become

$$[\phi(\mathbf{k},0), \phi^\dagger(\mathbf{k}',0)] = [\dot{\phi}(\mathbf{k},0), \dot{\phi}^\dagger(\mathbf{k}',0)] = 0 \qquad (12.4)$$

These relations hold if

$$(\mathbf{k} \,|\, \Omega_+\bar{\omega}^{\pm 1}\Omega_+^\dagger - \Omega_-\bar{\omega}^{\pm 1}\Omega_-^\dagger \,|\, \mathbf{k}') = 0$$

which is actually satisfied, as can be shown by use of (11.35). The last commutation relation

$$\begin{aligned} [\phi(\mathbf{k},0), \dot{\phi}^\dagger(\mathbf{k}',0)] &= \frac{i}{2}[(\mathbf{k} \,|\, \Omega_+\Omega_+^\dagger + \Omega_-\Omega_-^\dagger \,|\, \mathbf{k}') + 2U_b(\mathbf{k})U_b^*(\mathbf{k}')](2\pi)^{-3} \\ &= i\delta^3(\mathbf{k} - \mathbf{k}') \qquad (12.5) \end{aligned}$$

is also satisfied [see (11.27b)], but only because we introduced the extra bound-state terms in (12.1) together with the commutation rules (12.2). Of course, if there is no bound state, then we merely put $U_b = 0$, and (12.5) still holds.

12.2. Scattering. We briefly discussed the connection between the in and out fields in Chap. 11. We found that this relation is most easily expressed when the fields are developed in terms of angular-momentum eigenfunctions. This connection remains the same when the theory is quantized, as follows from (12.1) and the same equation in terms of ϕ^{out}. This pair of equations also shows that no transitions occur from (to) the continuum to (from) the bound state. We can rewrite (11.37) as

$$\begin{aligned} B_{lm}(k) &= S^{-1}A_{lm}(k)S = A_{lm}(k) \qquad \text{for } l \neq 0 \\ B_{00}(k) &= S^{-1}A_{00}(k)S = A_{00}(k)e^{2i\delta(k)} \\ B_{00}^{\dagger}(k) &= S^{-1}A_{00}^{\dagger}(k)S = A_{00}^{\dagger}(k)e^{-2i\delta(k)} \end{aligned} \tag{12.6}$$

and deduce that[1]

$$S = \exp\left[i\int A_{00}^{\dagger}(k)2\delta(k)A_{00}(k)\,dk\right] \tag{12.7}$$

From

$$N^{\text{in}} = \sum_{lm}\int dk\, A_{lm}^{\dagger}(k)A_{lm}(k)$$

and a similar expression for N^{out}, it follows that $N^{\text{in}} = N^{\text{out}}$; e.g., there is no production of real particles. Only that state which has an incoming particle with $l = 0$ differs from the corresponding state with an outgoing particle by a phase shift 2δ. An incoming plane wave, which is a mixture of eigenstates of **L**, will be an outgoing plane wave plus an outgoing spherical wave with an amplitude $(e^{2\,\delta} - 1)$. From this we can deduce the transition probabilities as in one-particle quantum mechanics. The scattering cross section (8.34) is the usual expression

$$\frac{d\sigma(k)}{d\Omega} = \frac{\sin^2\delta(k)}{k^2}$$

We shall now briefly comment on resonance scattering, which is an important feature of the pion-nucleon system. If we examine

$$\begin{aligned} \tan\delta(k) &= -\frac{\lambda}{4\pi}\frac{|\rho(k)|^2\,k}{D_1(k^2)} \\ &= -\frac{\lambda}{4\pi}\frac{|\rho(k)|^2\,k}{1+\lambda P\displaystyle\int\frac{d^3q}{(2\pi)^3}\frac{|\rho(q)|^2}{q^2-k^2}} \end{aligned} \tag{12.8}$$

for values of λ that are small enough so that D^{-1} has no poles, we see that if $\rho(k)$ goes to zero sufficiently rapidly for high momenta, then $\delta \to 0$ both as $k \to 0$ and as $k \to \infty$. For finite values of the momentum, δ assumes positive or negative values depending on whether λ is

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[1] We remind the reader that

$$[A_{lm}(k), A_{l'm'}^{\dagger}(k')] = \delta_{ll'}\delta_{mm'}\delta(k - k')$$

greater or less than zero. If λ is negative and its magnitude is increased, then, beyond a critical value, D_1 will become negative and D_1^{-1} will have two poles, as shown in Fig. 12.1. In such a case,[1] the phase shift will increase beyond 90°, at the momentum k_r defined by $D_1(k_r^2) = 0$. For momenta close to k_r we can expand $D_1(k^2)$,

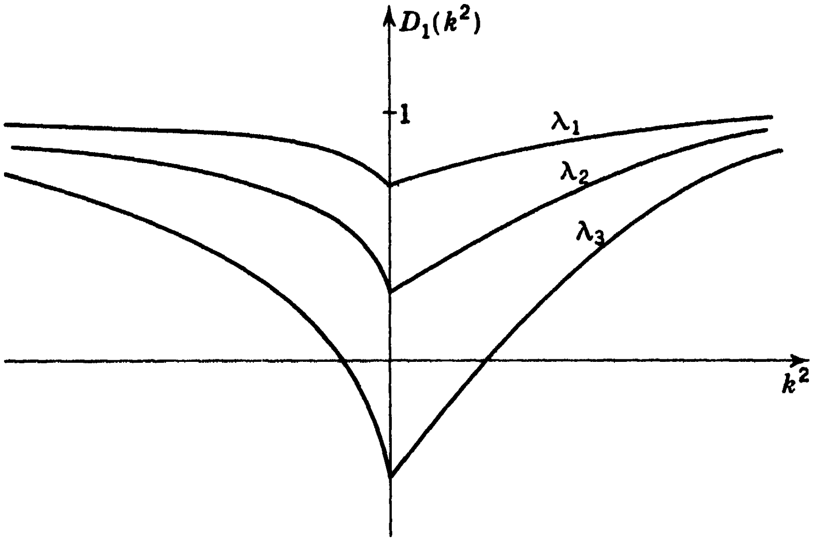

Fig. 12.1. Plot of $D_1(k^2)$ for three negative coupling constants, and $\lambda_1 > \lambda_2 > \lambda_3$.

$$D_1(k^2) \approx (k^2 - k_r^2) D_1'(k_r^2) + \cdots$$

where
$$D_1'(k_r^2) = \left[\frac{dD_1(x)}{dx}\right]_{x=k_r^2}$$

In this approximation the phase shift can be expressed as

$$\tan \delta(k) \approx \frac{\Gamma/2}{\omega_r - \omega} \tag{12.9a}$$

with
$$\Gamma = \frac{\lambda}{4\pi} \frac{k_r}{\omega_r} \frac{|\rho(k_r)|^2}{D_1'(k_r^2)} \tag{12.9b}$$

For the derivative of the phase shift with respect to the momentum, we find, near k_r,

$$\delta'(k) = \frac{\frac{1}{2}\Gamma k/\omega}{(\frac{1}{2}\Gamma)^2 + (\omega_r - \omega)^2}$$

---

[1] In the following we shall discuss the possibility that as the magnitude of the coupling strength is increased a resonance appears, followed by a bound state. This does not actually happen for S-wave scattering, but may occur for P waves with which we shall be concerned in the third part of this book.

which shows that δ' has a sharp maximum near ω_r for $\Gamma \ll \omega_r$.[¶] A very rapid increase of δ with k near a certain value of $k = k_r$, sometimes called resonance momentum and defined by $D_1(k_r^2) = 0$, indicates that incoming particles in this energy range tend to stay with the source and that the time required for the scattered particles to be emitted is long compared with the transit time of the particles past the source.[1] This can be seen by considering an incident gaussian (spherical) wave packet

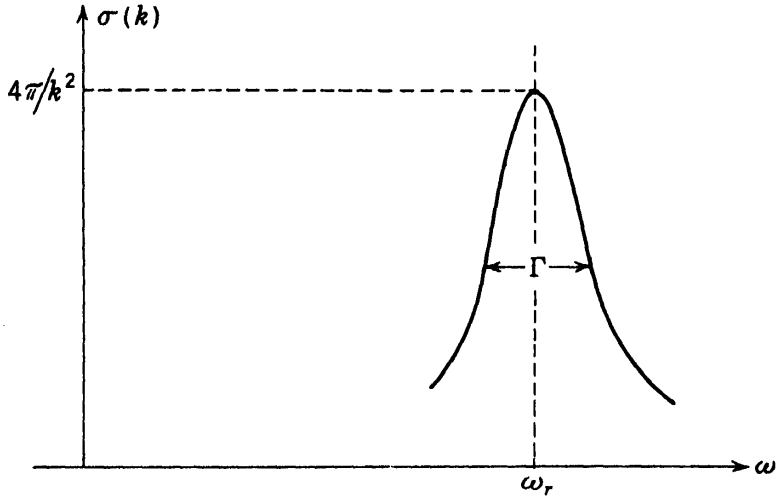

Fig. 12.2. Shape of the cross section near the resonant energy ω_r.

$$|U^{\text{in}}(r,t)|^2 = \left|\int_{-\infty}^{\infty} dk \exp\left[-\frac{(k-k_r)^2b^2}{2} - ikr - \frac{ik^2t}{2m}\right]\right|^2$$

$$\sim \exp - \left[\frac{(r-v_rt)^2}{b^2(t)}\right]$$

with $v_r = k_r/m$, $b^2(t) = b^2 + t^2/m^2b^2$. On expanding $\delta(k)$ near k_r, we find for the outgoing wave packet

$$|U^{\text{out}}(r,t)|^2$$

$$= \left|\int_{-\infty}^{\infty} dk \exp\left[-\frac{(k-k_r)^2b^2}{2} - ikr - \frac{ik^2t}{2m} + 2i(k-k_r)\delta'(k_r)\right]\right|^2$$

$$\sim \exp\left\{-\frac{[r - v_r(t+\Gamma^{-1})]^2}{b^2(t)}\right\}$$

where $\Gamma^{-1} = 2\delta'(k_r)/v_r$. This is also gaussian, but its peak has shifted by the time delay Γ^{-1}. Such a resonance is often called a "virtual" or "metastable" state with a lifetime Γ^{-1}. It dominates the scattering for particles with energies close to ω_r, since the phase shift is then close to 90° and the cross section is near its maximum. Inserting (12.9*a*) into the cross-section formula, we find

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[¶] This condition is not satisfied for S waves, but we shall continue the discussion below as if it were.

[1] For another discussion of resonance phenomena, see J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," chap. 8, John Wiley & Sons, Inc., New York, 1952.

$$\sigma(k) = 4\pi \frac{d\sigma(k)}{d\Omega} = \frac{\pi}{k^2} \frac{\Gamma^2}{(\Gamma/2)^2 + (\omega_r - \omega)^2} \tag{12.10}$$

The cross section is plotted for ω close to ω_r in Fig. 12.2. The expression (12.10) has a simple physical interpretation. The first factor is a geometrical one. It is the area perpendicular to the incident beam available to angular-momentum $l = 0$ waves of momentum k. The second term expresses the probability that a particle with energy ω is in the metastable state which has a width Γ and is centered around ω_r. Thus, (12.10) states that, for $k \approx k_r$, only those particles which have the right energy and angular momentum to form the metastable state are scattered.

If we increase $|\lambda|$ further, then the smaller of the values of k for which the phase shift δ goes through 90° comes closer to $k = 0$ and the time delay approaches ∞. This delay is reached for $\lambda = \lambda_c$ defined by $D(0) = 0$. If $|\lambda|$ is increased further, a bound state occurs. A bound state which occurs only for $\lambda < 0$ (attractive forces) and corresponds to a resonance at negative energies has as a consequence that δ is negative for low energies of the incoming particles. This happens for λ_3 of Fig. 11.1.

In the limit of high energies and for a finite source size, $\tan \delta(k)$ is given by its first term in an expansion in powers of λ (Born approximation),[1] since $D(\infty) = 1$. On the other hand, for $k \to 0$ there may be sizable corrections to the Born approximation, since the exact result differs from it by the factor $1/D_1(0)$. This may even become zero, as it does for a point source. This change of the cross section from its Born-approximation value has a simple physical interpretation if we consider the simplified version of electrodynamics[2] which includes only the $e^2A^2/2m_0$ term in (11.2) (and similarly for the γ_5 theory). We then find that $m = D(0)m_0$ is the total inertial mass of the electron, including the inertia of the electromagnetic field, and thus that the scattering cross sections in the low- and high-energy limits are e^2/m^2 and e^2/m_0^2, respectively. The effect of the higher approximations in e^2 (which represent virtual photons) is only an inertial one. For low frequencies the virtual photons move rigidly with the charge, and hence the relevant mass is m. For high frequencies they cannot follow, and the bare mass m_0 gives the scattering cross section. This simple result is typical for nonrelativistic quantum electrodynamics. In the relativistic theory virtual pairs of charged particles are created which change not only the effective mass of the electron but also the effective charge (vacuum polarization).

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[1] That the first term in the expansion does indeed correspond to the Born approximation, as it should, is shown, for example, by J. M. Blatt, *Phys. Rev.*, **72**:466 (1947).

[2] That **A** is a vector field does not introduce essential complications.

In our meson theoretic applications we shall find it convenient to introduce a renormalized interaction constant such that the exact expression for the phase shift, extrapolated to the unphysical energy $\omega = 0$, is formally equal to the Born approximation calculated with the renormalized coupling constant.[1] If this procedure, which may seem both unnecessary and arbitrary, is carried out, then we obtain a finite phase shift δ for finite momenta even for a point interaction. For a relativistic theory such a point interaction is required by Lorentz invariance; after renormalization, the results will be less sensitive to the shape of the source $\rho(r)$. In the present case the above procedure defines λ_r by

$$\lambda_r = \frac{\lambda}{D_1(-m^2)} = \frac{\lambda}{1 + \lambda P \displaystyle\int \frac{|\rho(q)|^2}{(2\pi)^3 \omega_q^2}\, d^3q} \tag{12.11a}$$

A fixed renormalized coupling constant λ_r [or, equivalently, $\lim\limits_{k\to 0} \delta(k)/k$] implies a certain dependence of λ on the source size. We noted at the end of the last chapter that $\lim\limits_{k\to 0} \delta(k)/k$ (or λ_r) goes to zero for a point source and a fixed value of λ. Thus, keeping λ_r fixed and finite, we obtain the following relation for λ in the limit of a point source (or a sufficiently small one):

$$\lambda = \frac{\lambda_r}{1 - \lambda_r P \displaystyle\int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_q^2}} \rightarrow - \frac{1}{P \displaystyle\int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega_q^2}} \rightarrow -0$$

This means that λ approaches zero from negative values, irrespective of the sign of λ_r. Although this leads to no difficulties here, with a single source, it creates a bound state with energy $<m$ for two or more sources. This implies that there is no state of lowest energy in the theory. In the Lee model, which we shall study next, the same kind of phenomenon occurs and can cause severe difficulties, even for a single source.

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[1] This procedure is often called "charge renormalization" because of its analogy to the same procedure for the charge e in quantum electrodynamics. See, e.g., S. S. Schweber, H. A. Bethe, and F. de Hoffmann, "Mesons and Fields," vol. I, p. 297, Row, Peterson & Company, Evanston, Ill., 1955.

In terms of the renormalized coupling constant, the phase shift is given by

$$\tan \delta(k) = -\frac{\lambda_r}{4\pi}\frac{k\,|\rho(k)|^2}{D_1^{(r)}(k^2)} \equiv -\frac{\lambda_r}{4\pi}\frac{k\,|\rho(k)|^2}{1+\lambda_r\omega^2 P\displaystyle\int\frac{d^3q\,|\,\rho(q)\,|^2}{(2\pi)^3\omega_q^2(q^2-k^2)}} \tag{12.11b}$$

As was pointed out above, this form has several virtues. For instance, the integral is weighted against high energies, where it is sensitive to the exact shape of $\rho(r)$, so that it even remains finite for a point source. The scattering may still exhibit a resonance about a particular energy ω_r, in the neighborhood of which Eqs. (12.9) and (12.10) are applicable if λ is replaced by λ_r and $D_1(k^2)$ by $D_1^{(r)}(k^2)$.

12.3. Energy Expressions in Terms of the Asymptotic Fields. We next turn to the problem of expressing H in terms of the asymptotic fields; as for the linear scalar coupling, we shall thus find the energy (or mass) renormalization, that is, the energy of the incoming vacuum. We shall see that the phase shift δ is directly involved in this computation.

By means of the field equation (8.11), we can write the Hamiltonian as[1]

$$\begin{aligned}
H &= \frac{1}{2}\int\left[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2 - \int\phi(\mathbf{r})V(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')\,d^3r'\right]d^3r \\
&= \int\frac{1}{2}\left[\dot{\phi}^\dagger(\mathbf{k})\dot{\phi}(\mathbf{k}) + \phi^\dagger(\mathbf{k})\omega^2\phi(\mathbf{k}) - \phi^\dagger(\mathbf{k})\int V(\mathbf{k}.\mathbf{k}')\phi(\mathbf{k}')\frac{d^3k'}{(2\pi)^3}\right]d^3k \\
&= \int\frac{1}{2}\left[\dot{\phi}^\dagger(\mathbf{k})\dot{\phi}(\mathbf{k}) - \phi^\dagger(\mathbf{k})\ddot{\phi}(\mathbf{k})\right]d^3k
\end{aligned} \tag{12.12a}$$

where we have used $\phi(\mathbf{k}) = \phi(\mathbf{k},0)$. Inserting (12.3) and computing the second time derivative of ϕ from (11.9*a*), we obtain, in matrix notation,

$$\begin{aligned}
H = {} & \frac{(2\pi)^3}{4}\left[A^\dagger\bar{\omega}^{\frac{1}{2}}\Omega_+^\dagger + A_b^\dagger U_b^*\omega_b^{\frac{1}{2}} - A\bar{\omega}^{\frac{1}{2}}\Omega_-^\dagger - A_bU_b\omega_b^{\frac{1}{2}}\right] \\
& \times\left[\Omega_+\bar{\omega}^{\frac{1}{2}}A + U_b\omega_b^{\frac{1}{2}}A_b - \Omega_-\bar{\omega}^{\frac{1}{2}}A^\dagger - U_b^*\omega_b^{\frac{1}{2}}A_b^\dagger\right] \\
& + \frac{(2\pi)^3}{4}\left[A^\dagger\bar{\omega}^{-\frac{1}{2}}\Omega_+^\dagger + A_b^\dagger U_b^*\omega_b^{-\frac{1}{2}} + A\bar{\omega}^{-\frac{1}{2}}\Omega_-^\dagger + A_bU_b\omega_b^{-\frac{1}{2}}\right] \\
& \times\left[\Omega_+\bar{\omega}^{\frac{3}{2}}A + U_b\omega_b^{\frac{3}{2}}A_b + \Omega_-\bar{\omega}^{\frac{3}{2}}A^\dagger + U_b^*\omega_b^{\frac{3}{2}}A_b^\dagger\right]
\end{aligned} \tag{12.12b}$$

---

[1] The Hamiltonian (12.12) has not been reordered according to the prescription (6.5), so that the usual zero-point energy E_0 of the noninteracting fields is included. The factor $(2\pi)^3$ arises because of our definition of matrix multiplication.

By making use of (11.36) and its hermitian conjugate as well as (11.29), we can simplify this expression to

$$H = \frac{(2\pi)^3}{2}(A^\dagger\tilde{\omega}A + A\tilde{\omega}A^\dagger + A_b^\dagger\omega_b A_b + A_b\omega_b A_b^\dagger)$$

$$= \int d^3k\, A^\dagger(k)A(k)\omega + A_b^\dagger A_b\omega_b + E \qquad (12.12c)$$

where E is formally equal to $\frac{1}{2}\int \omega\, d^3k + \frac{1}{2}\omega_b$. We shall shortly study the connection of this energy to that of the incoming vacuum. Aside from this constant, H is the energy of incoming and bound particles. Its spectrum in relation to that of the free field is shown in Fig. 12.3.

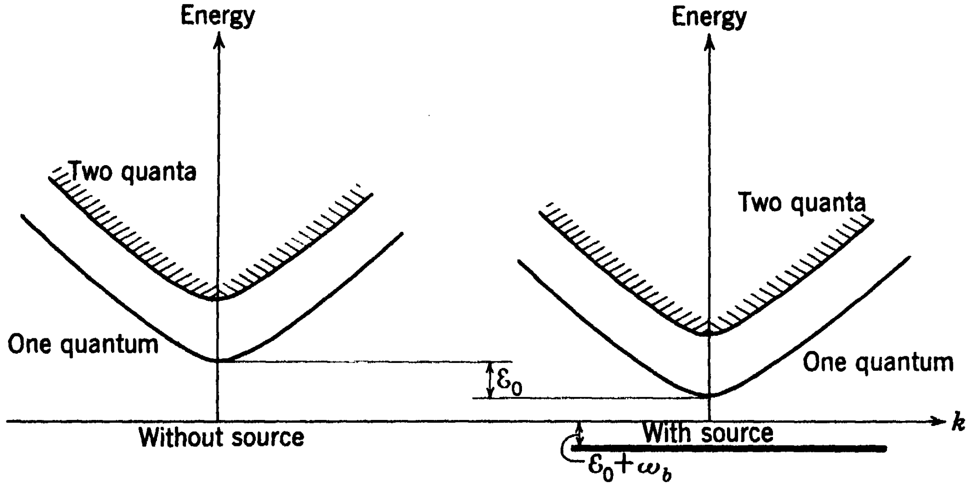

Fig. 12.3. Energy spectrum with and without a source. For the latter case the bound state is also shown.

The above development is possible only if ω_b is real, e.g., $|k_b^2| < m^2$, since the factor $\omega_b^{\frac{1}{2}}$ was assumed real. If $\omega_b^2 < 0$, then we are dealing with an oscillator with a repulsive force, $H = \frac{1}{2}(p^2 - \omega_b^2 q^2)$, and there is no discrete energy spectrum.[1] In particular, the energy eigenvalues then have no lower bounds, and no particle interpretation is possible. It is physically clear that a disaster starts to happen as soon as $|k_b^2| \rightarrow m^2$. When this critical momentum is reached, the total energy of this state is zero, and an infinitesimal amount of energy can create an infinity of particles in this bound state. These particles could appear as soon as the source is suddenly switched off. Thus, if λ is decreased adiabatically, then an instability will set in when it is negative and its magnitude exceeds a certain critical value determined by $k_b^2 = -m^2$.

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[1] Formally this resembles a "runaway solution," since the time dependence of the state has a term proportional to $e^{\omega_b t}$ when ω_b becomes imaginary. This resemblance is not accidental. Compare N. G. Van Kampen, *Physica*, **24**:545 (1958).

Since the Hamiltonian (12.12) was not reordered, it seems from (12.12*c*) that in the absence of a bound state the zero-point energy of the Hamiltonian is exactly equal to that for the free field. It thus appears that there is no energy renormalization for the pair theory, unlike the linear static case. In the notation of Chap. 9, we seem to obtain $\mathscr{E}_0 = 0$. However, the difference between two infinite (and therefore undetermined) terms may be different from zero even if they look the same, the result depending on how the limit is taken. To obtain a well-defined answer, we shall evaluate the energy difference between the physical vacuum states with and without a source by using a finite normalization sphere; only later shall we go to the continuum limit. We assume that no bound state is present. The boundary condition (5.9) tells us that the allowed values of k for the $l = 0$ part of the field are slightly different in the presence of the source because of the phase shift δ. Taking the usual standing-wave solutions, we find that if without interaction the possible values of k are k_n, defined by

$$\sin k_n R = 0$$

then with the source the boundary condition is

$$\sin\,[k'_n R + \delta(k'_n)] = 0$$

or (for $\delta k_n \ll k_n$)

$$k'_n = k_n - \frac{\delta(k_n)}{R} \tag{12.13}$$

This gives a change in the energy eigenvalues,

$$\omega'_n = \omega_n + \Delta\omega_n$$
$$\Delta\omega_n = -\frac{k_n}{\omega_n}\frac{\delta(k_n)}{R} \tag{12.14}$$

and hence a change in the zero-point energy,

$$\mathscr{E}_0 = \frac{1}{2}\sum_{k'} \omega' - \frac{1}{2}\sum_{k} \omega = -\frac{1}{2R}\sum_{k} \frac{k}{\omega}\,\delta(k) \tag{12.15a}$$

In the continuum limit this difference remains finite and is

$$\mathscr{E}_0 = -\frac{1}{2\pi}\int_0^\infty dk\,\frac{k}{\omega}\,\delta(k) \tag{12.15b}$$

Thus the energy shift or renormalization $\mathscr{E}_0$ due to the source is different from zero because the source brings the eigenfrequencies out of tune and thereby causes a change in the zero-point energy of the field. For the static source studied in Chap. 9, the origin of $\mathscr{E}_0$ was different. Because there was no scattering in that theory, terms of the type (12.15) did not contribute. The latter effect also causes forces between macroscopic bodies, since the possible electric frequencies depend on their distance. This amusing quantum effect has been verified experimentally.[1]

The relation (12.15) between the phase shift and the change of the energies in a finite volume shows why an attractive (repulsive) interaction gives a positive (negative) phase shift. It also suggests a way to estimate the magnitude of the phase shift. The interaction can roughly be pictured as changing[2] ω^2 by λ/V within the volume V where $\rho \neq 0$. The change of ω^2 for the low-lying states ($k \sim 1/R$) is λ/V times the probability of finding the particle within this volume. Hence

$$\delta\omega^2 = -\frac{k\delta(k)}{2R} \sim \frac{\lambda}{V}\frac{V}{R^3}$$

and

$$\delta(k) \sim -\frac{\lambda}{kR^2} \sim -\lambda k \tag{12.16}$$

in essential agreement with (12.8).

12.4. Virtual Particles. To conclude our discussion of the pair theory, we shall investigate the distribution of virtual particles. To this end we need the connection between the operators a_k [see (8.18)] which create the virtual particles and the A_k we have been using so far. This connection is easily obtained by comparing (8.18) and (12.3). With our matrix notation and in the absence of a bound state, we find that[3] (in our standard representation $\Omega_\pm^* = \Omega_\mp$)

$$a = M_1 A + M_2^* A^\dagger \qquad a^\dagger = M_1^* A^\dagger + M_2 A \tag{12.17}$$

$$\begin{aligned} M_1 &= \tfrac{1}{2}(\bar\omega^{\frac{1}{2}}\Omega_+\bar\omega^{-\frac{1}{2}} + \bar\omega^{-\frac{1}{2}}\Omega_+\bar\omega^{\frac{1}{2}})(2\pi)^{-\frac{3}{2}} \\ M_2 &= \tfrac{1}{2}(\bar\omega^{\frac{1}{2}}\Omega_+\bar\omega^{-\frac{1}{2}} - \bar\omega^{-\frac{1}{2}}\Omega_+\bar\omega^{\frac{1}{2}})(2\pi)^{-\frac{3}{2}} \end{aligned} \tag{12.18}$$

Note that the destruction operator for virtual particles is a mixture of destruction and creation operators for real particles. This is a typical relativistic effect; in the nonrelativistic limit $(k \mid \omega \mid k') \rightarrow m\delta(k-k')$, $M_2 = 0$, and the number-of-quanta operator $N^{\text{in}} = N$.¶ This means that there are no virtual quanta surrounding the source. Relativistically, on the other hand, we have a cloud of virtual particles even for the ground state ($N^{\text{in}} = 0$).

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[1] See M. J. Sparnaay, *Physica*, **24**:751 (1958).

[2] That this is the change of ω^2 rather than ω can also be seen from dimensional considerations.

[3] In this case a and A are also related by a unitary matrix Λ,

$$a = \Lambda A \Lambda^\dagger \qquad \Lambda\Lambda^\dagger = \Lambda^\dagger\Lambda = 1$$

¶ The consistency of the commutation relations for $a, a^\dagger$ and $A, A^\dagger$ requires $M_1 M_1^\dagger = 1$ when $M_2 = 0$.

The determination of the distribution of virtual particles is most easily done by comparison with a one-dimensional analogue, which also guided us for the same problem with linear coupling. The analogue for our present model is a simple harmonic oscillator, the frequency of which is changed from ω^2 to $\omega'^2 = \omega^2 + \delta\omega^2$. The former frequency corresponds to no source and operators a, $a^\dagger$, whereas the latter represents the energy in the presence of the source. From (2.3) and (2.4) we find that the quantities corresponding to (12.18) become

$$\begin{aligned} M_1 &= \frac{1}{2}\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}} + \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right] \\ M_2 &= \frac{1}{2}\left[\left(\frac{\omega}{\omega'}\right)^{\frac{1}{2}} - \left(\frac{\omega'}{\omega}\right)^{\frac{1}{2}}\right] \end{aligned} \tag{12.19}$$

From (12.17) and its hermitian conjugate we find

$$\begin{aligned} a \mid \text{in}, 0\rangle &= M^* a^\dagger \mid \text{in}, 0\rangle \\ M &= M_2 M_1^{-1} \end{aligned} \tag{12.20}$$

In the one-dimensional case, $M = -\delta\omega^2/(\omega + \omega')^2$ and is a number, whereas for the pair theory M is a matrix. To expand the eigenstates $|\text{in}, n\rangle$ of the Hamiltonian with $\delta\omega^2$ in terms of the eigenstates $|n)$ of the Hamiltonian without $\delta\omega^2$, we use (12.20), together with familiar tricks, to obtain

$$(n \mid 0\rangle = \left(0 \left| \frac{a^n}{(n!)^{\frac{1}{2}}} \right| \text{in}, 0\right\rangle = M^*\left(\frac{n-1}{n}\right)^{\frac{1}{2}} \left(0 \left| \frac{a^{n-2}}{[(n-2)!]^{\frac{1}{2}}} \right| \text{in}, 0\right\rangle \tag{12.21}$$

and by induction

$$\eta_n = \begin{cases} |(n \mid \text{in}, 0\rangle|^2 = \left(\dfrac{|M|}{2}\right)^n \dfrac{n!}{[(n/2)!]^2} \, |(0 \mid \text{in}, 0\rangle|^2 & \text{if } n \text{ is even} \\ 0 \quad \text{if } n \text{ is odd} \end{cases} \tag{12.22}$$

The quantity $|(0 \mid \text{in}, 0\rangle|^2$ is determined by the condition $\sum_0^\infty \eta_n = 1$. The same method works when M is a matrix in k space. Then (12.21) and (12.22) tell us that virtual particles are created only in pairs[1] and that the probability for finding an odd number of virtual particles is therefore zero. The simplest term is

$$\eta_{k_1,k_2} = |(k_1, k_2 \mid \text{in}, 0\rangle|^2 = \tfrac{1}{2}|(0 \mid \text{in}, 0\rangle|^2 \, |(k_1 \mid M \mid k_2)|^2 \tag{12.23}$$

---

[1] It is for this reason that the quadratic coupling discussed in this chapter and Chap. 11 is referred to as "pair theory."

Unfortunately, the explicit calculation of M, that is, the analytic inversion of M_2, is very involved, and we shall not enter into it.[1] As is to be expected, the wave function M of the virtual pair decreases exponentially with distance in ordinary space and is concentrated around ρ within a distance m^{-1}.

To get more tractable expressions, we turn to the calculation of expectation values of observables in the ground state. From (12.17) we immediately find that $\langle \text{in}, 0 \mid \phi(\mathbf{r},t) \mid \text{in}, 0\rangle = 0$, which also follows generally from the invariance of the Hamiltonian under $\phi \to -\phi$. More instructive are bilinear expressions, such as the mean number of virtual particles:

$$\begin{aligned}\bar{n} &= \left\langle \text{in}, 0 \middle| \int d^3k\, a^\dagger(\mathbf{k})a(\mathbf{k}) \middle| \text{in}, 0 \right\rangle \\ &= (2\pi)^3 \langle \text{in}, 0 \mid A(\mathbf{k})\, (\mathbf{k} \mid M_2^T M_2^* \mid \mathbf{k}')\, A^\dagger(\mathbf{k}') \mid \text{in}, 0\rangle \\ &= (2\pi)^3\, \mathrm{Tr}\, M_2^\dagger M_2 \end{aligned} \tag{12.24}$$

By means of (12.18) and (11.22) this can be rewritten as

$$\bar{n} = \tfrac{1}{4}\mathrm{Tr}\,(\bar{\omega}R_+^\dagger\bar{\omega}^{-1}R_+ + \bar{\omega}^{-1}R_+^\dagger\bar{\omega}R_+ - 2R_+^\dagger R_+)$$

Using our explicit expression for R, Eq. (11.23), we find

$$\begin{aligned}\bar{n} &= \frac{\lambda^2}{4}\int \frac{d^3k\, d^3k'}{(2\pi)^6}\, \frac{|\rho(k)|^2\, |\rho(k')|^2}{|D_+(k)|^2(k^2 - k'^2)^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2\right) \\ &= \frac{\lambda^2}{4}\int \frac{d^3k\, d^3k'}{(2\pi)^6}\, \frac{|\rho(k)|^2\, |\rho(k')|^2}{\omega\omega'(\omega + \omega')^2}\, |D_+(k^2)|^{-2} \end{aligned} \tag{12.25}$$

Thus the probability for finding one pair (or more) is essentially

$$\left[\frac{\lambda}{4}\int \frac{d^3k}{(2\pi)^3}\, \frac{|\rho(k)|^2}{\omega^2}\right]^2$$

a result which is similar to that for the linear coupling,

$$\frac{1}{2}\left[g\int \frac{d^3k}{(2\pi)^3}\, \frac{|\rho(k)|^2}{\omega^3}\right]^2$$

except that now the particles are not independent but are emitted in pairs. The amplitude for finding the field excited is of the order of $\bar{n}^{\frac{1}{2}}$ or

$$\frac{\lambda}{4}\int \frac{d^3k}{(2\pi)^3}\, \frac{|\rho(k)|^2}{\omega^2} = \frac{\lambda}{(8\pi^2)}\int_0^\infty \frac{k^2\, dk\, |\rho(k)|^2}{\omega^2}$$

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[1] It has been carried out by A. Klein and B. H. McCormick, *Phys. Rev.*, **98**:1428 (1955). These authors also derive a closed expression for the presence of more than one pair.

From (12.16) this is seen to be approximately $\int dk\ \delta\omega^2/\omega^2$ and is just the sum of the amplitudes for the various normal modes to be excited. For the single-oscillator analogue we find

$$\bar{n}^{\frac{1}{2}} = \frac{(\delta\omega)^2}{\omega^2} \qquad \text{if } \omega' \approx \omega$$

However, the energy of the virtual pairs is not simply related to $\mathscr{E}_0$ by a sort of virial theorem. Formally, however, it appears from (12.12) and (11.1) that in pair theory

$$\langle \text{in}, 0 \mid H_0 \mid \text{in}, 0\rangle = -\langle \text{in}, 0 \mid L' \mid \text{in}, 0\rangle$$

and $\mathscr{E}_0$ stems only from the change in the zero-point energy.

The probability for finding virtual pairs for a gaussian source distribution $\rho(r) \propto e^{-r^2/b^2}$ with a width $b \gg m^{-1}$ is of the order of $(\lambda/b^3m^2)^2$. Only a deep potential (large $|\lambda|$) with sharp edges will produce pairs of virtual particles in significant numbers. For a Coulomb potential $e/4\pi r$, only the steep part, $r \sim m^{-1}$, is effective in creating pairs. Approximating it there by a potential of our type ($\lambda \sim em^{-1}$), we find that the probability for virtual pairs $\sim e^2/4\pi \sim 1$ per cent. The virtual electron-positron pairs[1] in the Coulomb field actually give a measurable "vacuum-polarization" effect, since they are charged and change the Coulomb field for $r < m^{-1}$ ($\sim 10^{-11}$ cm) by about 1 per cent. Modern experimental techniques are capable of measuring such tiny effects with remarkable accuracy.[2]

Further Reading

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E. Arnous, *J. phys. radium*, **17**:107 (1956).

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[1] In this respect these pairs behave according to the above description, even though our calculation does not apply to them.

[2] See, e.g., W. E. Lamb and R. C. Retherford, *Phys. Rev.*, **72**:241 (1947).

CHAPTER 13

The Lee Model: States with $Q = \pm\frac{1}{2}$

13.1. Introduction. The Lee model consists of a field linearly coupled to a source which has one internal degree of freedom. This degree of freedom is not a classical quantity like position and has only two eigenvalues. As an example of such a quantity, we cite the charge of the nucleon (source) which has two eigenvalues distinguishing proton (p) and neutron (n). This degree of freedom, discussed in Chap. 7, is involved in the coupling which describes the elementary processes $n \leftrightarrow p + \pi^-$, if we choose to call the quanta of the field pions. In general, if the source has more than one degree of freedom, the problem gets so complicated that it cannot be solved explicitly. We shall encounter this situation in the last part of the book. However, in the Lee model it is assumed that there is no π^+, so that the process $p \leftrightarrow n + \pi^+$ cannot occur. In this case charge conservation limits the possibilities to such an extent that the problem becomes soluble. Although artificial, the model reflects important features of the pion-nucleon system. Formally, the problem has many similarities to the pair theory which we have discussed in the last two chapters. We shall lean heavily on the analogies in order to avoid repetitious formal manipulations.

To describe the degrees of freedom of the source, we have to introduce new dynamical variables. We define the operators $\tau_3(t)$, $\tau_+(t)$, $\tau_-(t) = \tau_+^\dagger(t)$ in terms of their effects on the bare nucleons at $t = 0$:

$$\begin{aligned} \tau_-(0)\,|\,p) &= |\,n) & \tau_-(0)\,|\,n) &= 0 \\ \tau_+(0)\,|\,n) &= |\,p) & \tau_+(0)\,|\,p) &= 0 \\ \tau_3(0)\,|\,p) &= |\,p) & \tau_3(0)\,|\,n) &= -\,|\,n) \end{aligned} \tag{13.1}$$

These operators are independent of time in the absence of an interaction and for equal masses of the nucleons (H_0 is then independent of the operators τ). Hence, in the absence of an interaction, the Hilbert space is the direct product of the usual one for the fields times a two-dimensional space. If the nucleon is represented by (p,n) in the latter, then the operators[1] τ are the 2 × 2 matrices

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13.2}$$

In the presence of an interaction the operators τ become time-dependent, and (13.2) is a representation of $\tau(0)$ in terms of bare particle states at $t = 0$. The interaction $H'(\tau)$ which generates the processes $n \leftrightarrow p + \pi^-$ between the bare particle states must be of the form

$$(\tau_+) \cdot (\text{creation operator}) + (\tau_-) \cdot (\text{destruction operator})$$

If we wish to forbid $p \leftrightarrow n + \pi^+$ for a pion described by a Klein-Gordon field, then a difficulty arises because all local[2] operators such as $\phi(\mathbf{r},t)$ contain both destruction and creation operators. Thus τ_+ should be multiplied not only by a creation operator for a π^- but also by a destruction operator for a π^+ [see (7.16)], and similarly for τ_-. However, for a Schrödinger field, $\psi(\mathbf{r})$ is a pure destruction operator and a suitable $H'(\mathbf{r})$ can easily be constructed. We base the model on the following Hamiltonian:[3]

$$H = \frac{1 - \tau_3(t)}{2} \mathscr{E}_0 + \int d^3r \Big\{ \nabla\psi^\dagger \cdot \nabla\psi \frac{1}{2m} + m\psi^\dagger\psi + g\rho(\mathbf{r})[\psi(\mathbf{r},t)\tau_-(t) + \psi^\dagger(\mathbf{r},t)\tau_+(t)] \Big\} \tag{13.3}$$

The $\mathscr{E}_0$ term is included to allow for a p-n mass difference; it is zero when it operates on a bare p state and $\mathscr{E}_0$ for a bare neutron. It will be used to renormalize the energy of the physical n state.

13.2. Commutation Relations and Equations of Motion. The commutation rules for the operators τ cannot be inferred from the canonical rules but can be deduced from the matrix representation (13.2). Since the operators $\tau(t)$ are created from $\tau(0)$ by the unitary transformation (2.18), they have the same commutation relations, which are

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[1] By τ without a subscript we mean any of the three operators τ_+, τ_-, τ_3.

[2] The Lee model [T. D. Lee, *Phys. Rev.*, **45**:1329 (1954)] usually consists of a non-local interaction, because it uses a relativistic theory for the boson. Such an interaction goes beyond the scope of our investigations. However, in its significant consequences, the model we shall use parallels the original one.

[3] This differs from the Schrödinger Hamiltonian of Chap. 4 by the source terms, the $\mathscr{E}_0$ term, and also by the addition of the term $m\psi^\dagger\psi$ to represent a rest energy of the field quanta. Note that g has the dimension of $L^{\frac{1}{2}}$.

$$\begin{aligned} [\tau_+(t), \tau_-(t)] &= \tau_3(t) \\ \tfrac{1}{2}[\tau_3(t), \tau_\pm(t)] &= \pm\tau_\pm(t) \end{aligned} \tag{13.4}$$

Of course, $\psi(t)$ and $\psi^\dagger(t)$ are assumed to commute with $\tau(t)$ at the same time,

$$[\psi(\mathbf{r},t), \tau(t)] = [\psi^\dagger(\mathbf{r},t), \tau(t)] = 0 \tag{13.5}$$

and the commutation rules for the operators ψ and $\psi^\dagger$ are the usual ones [see (4.22)]:

$$\begin{aligned} [\psi(\mathbf{r},t), \psi^\dagger(\mathbf{r}',t)] &= \delta^3(\mathbf{r} - \mathbf{r}') \\ [\psi(\mathbf{r},t), \psi(\mathbf{r}',t)] &= [\psi^\dagger(\mathbf{r},t), \psi^\dagger(\mathbf{r}',t)] = 0 \end{aligned} \tag{13.6}$$

With these commutation rules and $i\dot{\mathcal{O}} = [\mathcal{O},H]$ [see (2.19)], we derive for the equations of motion

$$-i\dot{\psi}^\dagger(\mathbf{r},t) = \left(m - \frac{\nabla^2}{2m}\right)\psi^\dagger(\mathbf{r},t) + g\rho(\mathbf{r})\tau_-(t) \tag{13.7a}$$

$$i\dot{\psi}(\mathbf{r},t) = \left(m - \frac{\nabla^2}{2m}\right)\psi(\mathbf{r},t) + g\rho(\mathbf{r})\tau_+(t) \tag{13.7b}$$

In momentum space these equations become

$$-i\dot{\psi}^\dagger(\mathbf{k},t) = w\psi^\dagger(\mathbf{k},t) + g\,\frac{\rho^*(\mathbf{k})}{(2\pi)^{3/2}}\,\tau_-(t) \tag{13.8a}$$

$$i\dot{\psi}(\mathbf{k},t) = w\psi(\mathbf{k},t) + g\,\frac{\rho(\mathbf{k})}{(2\pi)^{3/2}}\,\tau_+(t) \tag{13.8b}$$

with $$w = m + \frac{k^2}{2m} \qquad \psi^\dagger(\mathbf{k},t) = \int \psi^\dagger(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}\,\frac{d^3r}{(2\pi)^{3/2}}$$

$$\psi(\mathbf{k},t) = \int \psi(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}}\,\frac{d^3r}{(2\pi)^{3/2}} \qquad \text{and} \qquad \rho(\mathbf{k}) = \int \rho(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3r$$

These differential equations can be converted to integral ones in a manner analogous to Eqs. (8.4) to (8.10). By means of the retarded Green's function

$$\Delta^{\text{ret}\dagger}(\mathbf{k},t) = \frac{1}{2\pi}\int dK_0\,\frac{e^{iK_0t}}{K_0 - w - i\epsilon} = \begin{cases} 0 & \text{for } t < 0 \\ ie^{iwt} & \text{for } t > 0 \end{cases}$$

and its hermitian conjugate, we obtain[1]

$$\psi^\dagger(\mathbf{k},t) = \psi^{\dagger\text{in}}(\mathbf{k},t) + ig\int_{-\infty}^{t} dt'\,\tau_-(t')\,e^{iw(t-t')}\,\frac{\rho^*(\mathbf{k})}{(2\pi)^{\frac{3}{2}}} \tag{13.9a}$$

$$\psi(\mathbf{k},t) = \psi^{\text{in}}(\mathbf{k},t) - ig\int_{-\infty}^{t} dt'\,\tau_+(t')\,e^{-iw(t-t')}\,\frac{\rho(\mathbf{k})}{(2\pi)^{\frac{3}{2}}} \tag{13.9b}$$

For the operators τ we find

$$\begin{aligned} -i\dot{\tau}_-(t) &= \mathscr{E}_0\tau_-(t) + g\int d^3r\;\rho(\mathbf{r})\psi^\dagger(\mathbf{r},t)\tau_3(t) \\ &= \mathscr{E}_0\tau_-(t) + g\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\;\rho(\mathbf{k})\psi^\dagger(\mathbf{k},t)\tau_3(t) \end{aligned} \tag{13.10a}$$

$$i\dot{\tau}_+(t) = \mathscr{E}_0\tau_+(t) + g\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\;\rho^*(\mathbf{k})\psi(\mathbf{k},t)\tau_3(t) \tag{13.10b}$$

and, by means of (13.7*a*) and (13.7*b*),

$$\begin{aligned} \tfrac{1}{2}\dot{\tau}_3(t) &= ig\int d^3r\;\rho(\mathbf{r})[\psi(\mathbf{r},t)\tau_-(t) - \psi^\dagger(\mathbf{r},t)\tau_+(t)] \\ &= \frac{\partial}{\partial t}\int d^3r\;\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t) \end{aligned} \tag{13.11}$$

The first two equations can also be written in integral form, but we shall defer this for a while. The last equation expresses charge conservation, since it says

$$\dot{Q} = 0 \qquad Q = \tfrac{1}{2}\tau_3 - \int d^3r\;\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) = \tfrac{1}{2}\tau_3 - \int d^3k\;\psi^\dagger(\mathbf{k})\psi(\mathbf{k}) \tag{13.12}$$

It is only the total charge (i.e., the charge of the source together with that of the field) which is conserved. The charge of the source separately, or of the field alone, is not conserved. We note that the operator Q has half-integral eigenvalues $\leq\frac{1}{2}$ corresponding to the following bare particle states:[2]

Q	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\cdots$
Particle	p	n	$n\pi^-$	$\cdots$
		$p\pi^-$	$p\pi^-\pi^-$	$\cdots$

---

[1] By this time the reader should be familiar with the adiabatic principle, which justifies the physical interpretation of

$$\psi^{\text{in}\dagger}(\mathbf{k},t) = \psi^{\text{in}\dagger}(\mathbf{k},0)\,e^{iwt} \qquad\text{and}\qquad \psi^{\text{in}}(\mathbf{k},t) = \psi^{\text{in}}(\mathbf{k},0)\,e^{-iwt}$$

The w which appears in the exponent here and elsewhere and the w which is used in the denominator of the last equation of page 128, for example, are to be interpreted as the same symbol.

[2] The actual electric charge is $Q + \frac{1}{2}$ and has integral eigenvalues. Q is the third component of the total isospin and will be used in the last part of the book. The electric charge of the nucleon is $\frac{1}{2}(1 + \tau_3)$, and that of the meson is $-\int d^3r\,\psi^\dagger(\mathbf{r})\psi(\mathbf{r})$, because we are dealing with π^- mesons.

The equations of motion (13.7) to (13.10) are still too complex to be integrated in closed form. However, when applied to physical particle states (eigenstates of Q and H), they simplify further because in the present model these states are mixtures of states with n and $n + 1$ mesons.

13.3. Physical Nucleons. The bare proton state[1] $|p)$ and the physical one are identical,

$$\begin{aligned} | p\rangle &= | p) = | \text{in}, p\rangle = | \text{out}, p\rangle \\ \tau_+(t) | p\rangle &= \psi(\mathbf{r},t) | p\rangle = 0 \\ \tau_3(t) | p\rangle &= | p\rangle \end{aligned} \tag{13.13}$$

and both are eigenstates of H,

$$H | p\rangle = 0 \tag{13.14}$$

This stems from the fact that there is no other state belonging to the same eigenvalue of Q. If we apply the equations of motion to $| p\rangle$, we can replace τ_3 by 1 and obtain two coupled linear equations. In the following we shall first seek the solution of these equations in terms of the initial values of the operators. Then we shall see that this solution also satisfies the correct commutation relations if we postulate that the incoming operators have the same commutation properties as the ones without interaction. To formulate this conveniently, we denote with a bar those operators which are multiplied from the right with a projection operator onto the proton state. This means that barred equations are true only when applied to the proton state. In particular, $\bar{\tau}_3(t) = 1$, and we get for the Fourier transform of (13.9) and (13.10) with respect to time[2]

$$\begin{aligned} &\bar{\psi}^\dagger(\mathbf{k},K_0) = \int \bar{\psi}^\dagger(\mathbf{k},t)e^{-iK_0t}\, dt = \bar{\psi}^{\dagger\text{in}}(\mathbf{k},K_0) - \frac{g}{(2\pi)^{\frac{3}{2}}} \frac{\rho^*(\mathbf{k})\bar{\tau}_-(K_0)}{w - K_0 + i\epsilon} \\ &\bar{\psi}(\mathbf{k},K_0) = 0 \\ &(K_0 - \mathscr{E}_0)\bar{\tau}_-(K_0) = g \int \frac{d^3k\ \rho(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}\, \bar{\psi}^\dagger(\mathbf{k},K_0) \end{aligned} \tag{13.15}$$

If we were to eliminate $\bar{\tau}_-(K_0)$ from the third equation and put it into the first, we should obtain an equation almost identical with (10.7) of the pair theory. Again, the values of K_0 for which (13.15) has solutions will reflect the eigenvalue spectrum of H for the states generated by the application of τ_- and $\psi^\dagger$ to $| p)$. Since both τ_- and $\psi^\dagger$ create a state with $Q = -\frac{1}{2}$, we expect a discrete state corresponding to the physical neutron and a continuum of $p + \pi^-$ states beginning at $K_0 = m$. If we denote the energy of the physical neutron by ΔM, we can adjust $\mathscr{E}_0$ so that $\Delta M < m$, in which case the neutron is stable. The spectrum for this case appears in Fig. 13.1. Like the bound state in pair theory, the physical neutron must correspond to a solution of (13.15) with $K_0 = \Delta M < m$, where $\psi^{\dagger\text{in}} = 0$.[¶] Hence we have, for $K_0 = \Delta M$,

---

[1] Such a state corresponds to a bare vacuum state $| 0)$ in our old notation. The labels $| p)$ and $| n)$ serve to distinguish between the internal degrees of freedom.

[2] In the usual manner we define $\tau_-(K_0) \equiv \int \tau_-(t)\, e^{-iK_0t}\, dt$.

$$(\Delta M - \mathscr{E}_0)\bar{\tau}_-(\Delta M)$$

$$= g^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(\mathbf{k})|^2}{\Delta M - w}\, \bar{\tau}_-(\Delta M)$$

$$\text{or}\quad \Delta M = -g^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(\mathbf{k})|^2}{w - \Delta M} + \mathscr{E}_0 < \mathscr{E}_0 \qquad (13.16)$$

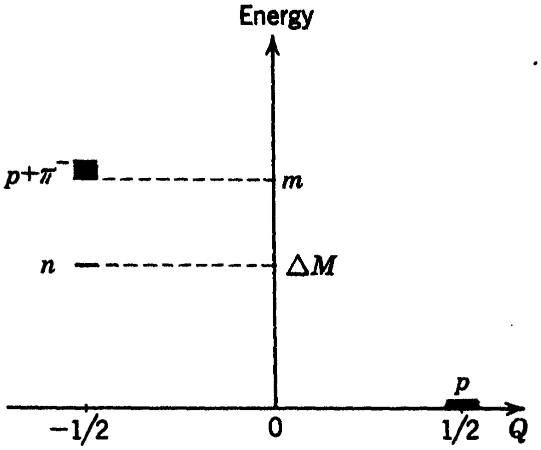

Fig. 13.1. Eigenvalue spectrum of the Hamiltonian (13.3) for the states of $Q = \pm\frac{1}{2}$.

Before the source is turned on, the energy of the neutron is $\mathscr{E}_0$. Since $w > m$, and it was assumed that $\Delta M < m$, (13.16) shows that the interaction lowers the energy of the neutron, as in the case of a static source.

13.4. Scattering States. The solutions of (13.15) with $K_0 > m$ can be found as in the pair theory:

$$\bar{\tau}_-(K_0)\left[K_0 - \mathscr{E}_0 + g^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(\mathbf{k})|^2}{w - K_0 + i\epsilon}\right] = g \int \frac{d^3k\, \rho(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}\, \bar{\psi}^{\dagger\text{in}}(\mathbf{k},K_0)$$

Because of (13.16), we can eliminate the unobservable $\mathscr{E}_0$ from the left-hand side, the bracket of which can be written as[1]

$$K_0 - \Delta M + g^2 \int \frac{d^3k}{(2\pi)^3} |\rho(\mathbf{k})|^2 \left(-\frac{1}{w - \Delta M} + \frac{1}{w - K_0 + i\epsilon}\right)$$

$$= (K_0 - \Delta M) D_-(K_0)$$

$$\equiv (K_0 - \Delta M)\left[1 + g^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(\mathbf{k})|^2}{(w - K_0 + i\epsilon)(w - \Delta M)}\right] \qquad (13.17)$$

[¶] This is interesting, because one tends to think that a physical particle is something different from a bound state of bare particles. However, our example shows that so far there is no fundamental difference between a "compound particle" and a physical one.

[1] The factor $i\epsilon$ plays no role in $w - \Delta M$ so long as ΔM is less than m, as has been assumed. This follows because $\psi^{\dagger\text{in}}(\mathbf{k},t)$ has the free-field time dependence and $\psi^{\dagger\text{in}}(\mathbf{k},K_0) \propto \delta(K_0 - w)$.

$D_{-}(K_0)$ is the boundary value of an analytic function of the complex variable $K_0 = z = x + iy$. As in Chap. 11, we find that $D(z)$ does not have any zeros for $x < m$. In fact, $D(-\infty + i0) = 1$ and $dD(x)/dx > 1$. Since $\psi^{\text{in}}(\mathbf{k},K_0)$ is different from zero only when $K_0 = m + k^2/2m > m$, we find that (13.15) has a nonzero solution for all $K_0 > m$ and for $K_0 = \Delta M$. The contribution from the latter point will be denoted by τ_{-}^{in}; to satisfy the commutation relations, we shall have to multiply it by a normalization factor $Z^{\frac{1}{2}}$, the meaning and value of which will be determined shortly. We can summarize our findings by

$$\bar{\tau}_{-}(K_0) = Z^{\frac{1}{2}}\bar{\tau}_{-}^{\text{in}}\delta(K_0 - \Delta M) + g\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{\rho(\mathbf{k})\bar{\psi}^{\dagger\text{in}}(\mathbf{k},K_0)}{(K_0 - \Delta M - i\epsilon)D_{-}(K_0)} \tag{13.18}$$

where the $-i\epsilon$ in the denominator corresponds to integrating (13.10*a*) with a retarded Green's function.[1] The solution for $\bar{\psi}^{\dagger}$ can be found as in Chap. 11, by making use of the Green's function and introducing the four-dimensional Fourier transform $\bar{\psi}^{\dagger}(\mathbf{k},K_0)$. We shall not repeat this procedure here but shall merely give the results:[2]

$$\bar{\psi}^{\dagger}(\mathbf{k},t) = \frac{U_b(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}\,\bar{\tau}_{-}^{\text{in}}(t) + (\mathbf{k}\,|\,\Omega_{-}\,|\,\mathbf{k}')\bar{\psi}^{\dagger\text{in}}(\mathbf{k}',t) \tag{13.19a}$$

$$U_b(\mathbf{k}) = Z^{\frac{1}{2}}g\,\frac{\rho^*(\mathbf{k})}{\Delta M - w} \tag{13.19b}$$

$$\begin{aligned}(\mathbf{k}\,|\,\Omega_{-}\,|\,\mathbf{k}') &= \delta^3(\mathbf{k} - \mathbf{k}')(2\pi)^3 + g^2\,\frac{\rho^*(\mathbf{k})\rho(\mathbf{k}')D_{-}^{-1}(w')}{(w' - w - i\epsilon)(w' - \Delta M)} \\ &= (\mathbf{k}\,|\,\Omega_{+}^{*}\,|\,\mathbf{k}')\end{aligned} \tag{13.19c}$$

We note that τ_{-}^{in} plays the same role as the bound-state operator A_b in the pair theory, but U_b is not normalized to 1, as we shall see presently. Similarly, the Fourier transform of (13.18) gives

$$\bar{\tau}_{-}(t) = Z^{\frac{1}{2}}\bar{\tau}_{-}^{\text{in}}(t) + g\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\,\frac{\rho(\mathbf{k})D_{-}^{-1}(w)}{w - \Delta M}\,\bar{\psi}^{\dagger\text{in}}(\mathbf{k},t) \tag{13.20}$$

remembering that ${}^{\dagger\text{in}}(\mathbf{k},K_0) \propto \delta(w - K_0)$.

13.5. Completeness. We are now in a position to demonstrate explicitly that the commutation relations (13.4) to (13.6) hold in the subspace $Q = \frac{1}{2}$ if we assume that τ^{in} and ψ^{in} satisfy the commutation rules of the operators without interaction and if $Z = D^{-1}(\Delta M)$, as shown below. In a manner similar to Eqs. (12.3) to (12.5), we calculate[1]

---

[1] It should be noted that the homogeneous equation for our (exact) solution is $i\dot{\tau}_{-} = -\Delta M\,\tau_{-}$ rather than $i\dot{\tau}_{-} = -\mathscr{E}_0\tau_{-}$.

[2] Matrix multiplication, defined as in Chap. 11, is implied here.

$$\langle p \mid [\tau_+(0), \tau_-(0)] \mid p\rangle = \langle p \mid \left[Z^{\frac{1}{2}}\tau_+^{\text{in}} + g\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{\rho^*(\mathbf{k})\psi^{\text{in}}(\mathbf{k})}{(w-\Delta M)D_+(w)} \right]$$

$$\times \left[g\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{\psi^{\dagger\text{in}}(\mathbf{k})\rho(\mathbf{k})}{(w-\Delta M)D_-(w)} + \tau_-^{\text{in}}Z^{\frac{1}{2}} \right] |p\rangle$$

$$= Z + \mathop{\mathbf{S}}_{\mathbf{k}} g^2 \frac{|\rho(\mathbf{k})|^2}{(w-\Delta M)^2\,|D_-(w)|^2} = Z - \frac{1}{2\pi i}\int_C \frac{dz}{(z-\Delta M)D(z)} \tag{13.21}$$

where C is a contour similar to part of Fig. 11.2 and is shown in Fig. 13.2. We note that $D^{-1}(z)$ is analytic save for a cut on the real axis from m to ∞. But for $|z| \to \infty$ it tends to 1, so that an integral over an infinite circle gives

$$\frac{1}{2\pi i}\int \frac{dz}{z} = 1$$

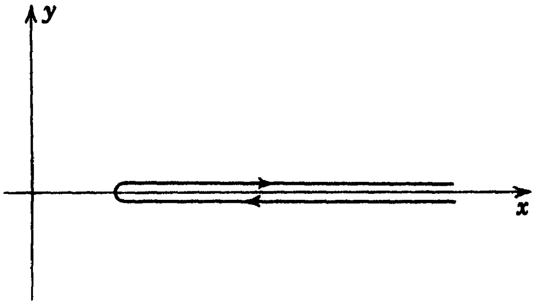

Fig. 13.2. Integration contour C defined by Eq. (13.21).

Hence, on closing the contour by a circle at ∞ and denoting the closed contour integral by $\oint$, we get

$$(13.21) = Z - \frac{1}{2\pi i}\oint \frac{dz}{(z-\Delta M)D(z)} + 1 = Z - \frac{1}{D(\Delta M)} + 1$$

$$= \langle p \mid \tau_3 \mid p\rangle = 1 \tag{13.22}$$

which implies that we have to put

$$Z = \frac{1}{D(\Delta M)} = \frac{1}{1 + g^2\displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{|\rho(\mathbf{k})|^2}{(w-\Delta M)^2}} \tag{13.23}$$

By similar arguments, we prove that

$$U_b(\mathbf{k})U_b^*(\mathbf{k}') + (\mathbf{k} \mid \Omega_- \mid \mathbf{q})(\mathbf{q} \mid \Omega_-^\dagger \mid \mathbf{k}') = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \tag{13.24}$$

a formula familiar to us from pair theory. Consequently,

$$\langle p \mid [\psi(\mathbf{k}), \psi^\dagger(\mathbf{k}')] \mid p\rangle = \delta^3(\mathbf{k}-\mathbf{k}') \tag{13.25}$$

---

[1] It is interesting to note that $\tau_-^{\text{in}}(t)$ does not commute with $\psi(\mathbf{r},t)$. Physically, this arises because τ_-^{in} creates a physical neutron, and the latter is an extended structure which interferes with the meson cloud even outside the source. This is in contradistinction to bare particles, the pointlike nature of which is expressed by the canonical commutation rules.

is satisfied. Also,

$$\langle p \mid [\tau_+, \psi^\dagger(\mathbf{k})] \mid p\rangle$$

$$= g\,\frac{\rho^*(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}\left[\frac{D_+^{-1}(w)}{w-\Delta M}+\frac{D^{-1}(\Delta M)}{w-\Delta M}-\frac{1}{2\pi i}\int\frac{dz\,D^{-1}(z)}{(z-\Delta M)(z-w)}\right]=0 \quad (13.26)$$

can be checked.[1]

Having established a solution of the equations of motion and the commutation relations in this sector of Hilbert space, we shall next turn to a discussion of the physical significance of the quantities encountered there. The physical neutron state is created by applying τ_-^{in} to the proton state

$$\mid \text{in}, n\rangle = \tau_-^{\text{in}} \mid p\rangle \qquad \langle \text{in}, n \mid = \langle p \mid \tau_+^{\text{in}}$$

Since τ_-^{in} has a time dependence $\sim e^{it\Delta M}$, it follows that $\mid \text{in}, n\rangle$ is an eigenstate[2] of H with eigenvalue ΔM, as we saw earlier. The factor Z gives the probability of finding a bare neutron in a physical one,

$$(n \mid \text{in}, n\rangle = \langle p \mid \left[Z^{\frac{1}{2}}\tau_+^{\text{in}} + g\int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}\,\frac{\rho(\mathbf{k})\psi^{\text{in}}(\mathbf{k})}{(w-\Delta M)D_+(w)}\right]\tau_-^{\text{in}} \mid p\rangle = Z^{\frac{1}{2}} \quad (13.27)$$

and $U_b(\mathbf{k})$ is the wave function of the virtual pion when the neutron is dissociated into $p + \pi^-$:

$$(p + \pi_{\mathbf{k}}^- \mid \text{in}, n\rangle = (p \mid \psi(\mathbf{k}) \mid \text{in}, n\rangle$$

$$= \langle p \mid \left[\frac{U_b(\mathbf{k})\tau_+^{\text{in}}}{(2\pi)^{\frac{3}{2}}} + (\mathbf{k} \mid \Omega_+ \mid \mathbf{k}')\psi^{\text{in}}(\mathbf{k}')\right]\tau_-^{\text{in}} \mid p\rangle = \frac{U_b(\mathbf{k})}{(2\pi)^{\frac{3}{2}}} \quad (13.28)$$

With some calculations we find that this wave function has the following properties. $U_b(\mathbf{k})$ is normalized, as it should be, to the probability of finding the neutron dissociated:[3]

$$\int\frac{d^3k}{(2\pi)^3}|U_b(\mathbf{k})|^2 = 1 - Z \quad (13.29)$$

The form of U_b, Eq. (13.19*b*), shows that the virtual pions stay near the source within a distance $(m - \Delta M)^{-1}$. Furthermore, for these virtual pions the virial theorem is valid in the same form as for the static source. For instance, for $\Delta M = 0$, H^{int}, the part of the Hamiltonian (13.3) proportional to g, and $H^{\text{meson}} = \int d^3r\, \nabla\psi^\dagger \cdot \nabla\psi/2m$, we have

---

[1] The proof of the commutation relations in the other parts of Hilbert space is not trivial and will not be carried out.

[2] This can also be checked directly with (13.3) and the analysis of $\mid \text{in}, n\rangle$ in terms of bare particle states, which we shall now discuss.

[3] Note that $0 \leq Z \leq 1$ is required.

$$0 = \langle \text{in}, n \mid H \mid \text{in}, n\rangle = \langle \text{in}, n \mid H^{\text{meson}} \mid \text{in}, n\rangle + \langle \text{in}, n \mid H^{\text{int}} \mid \text{in}, n\rangle + \langle \text{in}, n \mid \frac{1-\tau_3}{2} \mid \text{in}, n\rangle \mathscr{E}_0 \tag{13.30}$$

and
$$\langle \text{in}, n \mid H^{\text{meson}} \mid \text{in}, n\rangle = \mathop{S}_{\mathbf{k}} |U_b(\mathbf{k})|^2 w = Zg^2 \mathop{S}_{\mathbf{k}} \frac{|\rho(\mathbf{k})|^2}{w} = \langle \text{in}, n \mid \mathscr{E}_0 \frac{1-\tau_3}{2} \mid \text{in}, n\rangle$$

and hence
$$\langle \text{in}, n \mid H^{\text{int}} \mid \text{in}, n\rangle = -2\langle \text{in}, n \mid H^{\text{meson}} \mid \text{in}, n\rangle$$

The fact that the commutation relations for the operators ψ, $\psi^\dagger$, and τ are satisfied shows that the states we found with $Q = -\frac{1}{2}$ form a complete set. Hence there is, for instance, only one discrete state for the system of a p and a π^-, namely,

$$|\text{in}, n\rangle = Z^{\frac{1}{2}} \mid n) + \int \frac{d^3k\, U_b(\mathbf{k})\psi^\dagger(\mathbf{k}) \mid p)}{(2\pi)^{\frac{3}{2}}} \tag{13.30a}$$

This has its formal origin in the fact that $(K_0 - \Delta M)D_\pm(K_0)$ has only one zero for $K_0 = \Delta M$, as we saw earlier. By increasing ΔM, the range of $U_b(\mathbf{r})$ becomes larger and reaches infinity for $\Delta M = m$. When this value is reached, the neutron becomes unstable[1] against decay into a proton and a pion, and the sole discrete state for $Q = -\frac{1}{2}$ disappears. The commutation relations are then satisfied with $\bar{\tau}_-^{\text{in}} = 0$, and we shall see that the physical neutron is then only a resonance in the pion-proton scattering; the wave function of the pion for the scattering states $|\text{in}, p + \pi^-\rangle$ is singular in $\mathbf{k}$ space, which means that there are pions at an infinite distance from the nucleon.

13.6. The Phase Shift. To end this chapter, we investigate the scattering in the states $p + \pi^-$. For this purpose we must relate $\psi^{\dagger\text{in}}$ to $\psi^{\dagger\text{out}}$. The preceding development could have been carried out equally well with $\psi^{\dagger\text{out}}$ and $\bar{\tau}_-^{\text{out}}$, which is identical with $\bar{\tau}_-^{\text{in}}$:

$$\bar{\tau}_-^{\text{out}}(t) = e^{i\Delta Mt}\bar{\tau}_-^{\text{out}}(0) = \bar{\tau}_-^{\text{in}}(t) \tag{13.31}$$

$$\text{in}, n\rangle = |\text{out}, n\rangle \equiv |n\rangle$$

Hence we have

$$(\mathbf{k} \mid \Omega_- \mid \mathbf{k}')\bar{\psi}^{\dagger\text{in}}(\mathbf{k}') = (\mathbf{k} \mid \Omega_+ \mid \mathbf{k}')\bar{\psi}^{\dagger\text{out}}(\mathbf{k}') \tag{13.32}$$

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[1] For $\Delta M > m$ the model provides a field theoretic description of α decay if the various particles are suitably renamed.

As in pair theory, for a spherical source only the angular-momentum $l = 0$ parts of $\bar{\psi}^{\dagger\text{out}}$ and $\bar{\psi}^{\dagger\text{in}}$ differ by a phase shift $e^{-2i\delta}$,

$$\bar{\psi}^{\dagger\text{out}} = e^{-2i\delta}\bar{\psi}^{\dagger\text{in}}$$

and by analogy with Eqs. (11.31) to (11.38) we find

$$\tan\delta(k) = -\frac{D(w)}{D_1(w)} = \frac{-g^2 km\,|\rho(k)|^2}{2\pi(w-\Delta M)\left[1 + g^2 P\displaystyle\int \frac{d^3k'\,|\rho(k')|^2}{(2\pi)^3(w'-\Delta M)(w'-w)}\right]} \tag{13.33}$$

where

$$D_+(w) = D_1(w) + iD(w) = \left[1 + g^2 P \mathop{S}_{\mathbf{k}'} \frac{|\rho(k')|^2}{(w'-w)(w'-\Delta M)}\right] + i\left[\frac{g^2 m}{2\pi}\,\frac{k\,|\rho(k)|^2}{(w-\Delta M)}\right]$$

For $\Delta M < m$ and for $k \to 0$, we have $\delta(k) < 0$, as in pair theory with one bound state.[1] If this resonance at negative energies is raised to positive energies, $\Delta M > m$, then $\lim_{k\to 0} \delta(k)/k$ becomes positive, resembling a weak attractive potential. In this case the neutron becomes unstable with a lifetime[2] Γ^{-1}. This lifetime is defined by (12.9*a*), and we find

$$\Gamma = \frac{g^2 k_r m\,|\rho(k_r)|^2}{2\pi\left[1 + g^2 P\displaystyle\int \frac{d^3k'}{(2\pi)^3}\,\frac{|\rho(k')|^2}{(w'-\Delta M)^2}\right]} \tag{13.34a}$$

with

$$\frac{k_r^2}{2m} + m = \Delta M = w_r \tag{13.34b}$$

As might be anticipated from the similarity to pair theory, we can define a renormalized coupling constant, so that for $w = \Delta M$ the phase

[1] This is characteristic of bound states. See, e.g., J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," chap. 2, John Wiley & Sons, Inc., New York, 1952.

[2] Under certain conditions

$$(n(t)\mid n(0)) = (p \mid \tau_+(t)\tau_-(0) \mid p)$$

shows a behavior $\sim e^{-\Gamma t/2}$ [V. Glaser, G. Källen, *Nuclear Phys.*, **2**:706 (1956–1957)]. If $|\rho(k)|^2$ can be analytically continued into the complex plane, so that D can also be continued beyond the cut, then we find a pole on the second Riemannian sheet at a complex energy $\Delta M + i\Gamma$, corresponding to the unstable neutron state.

shift is given by the Born approximation with the renormalized coupling constant. We then find

$$g_r^2 = g^2 \frac{1}{1 + g^2 P \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(k)|^2}{(w - \Delta M)^2}} = Zg^2 \tag{13.35}$$

$$g_r = Z^{\frac{1}{2}} g$$

and

$$\tan \delta(k) = \frac{-g_r^2 k m\, |\rho(k)|^2}{2\pi(w - \Delta M)\left[1 + \frac{g_r^2(w - \Delta M)}{(2\pi)^3} P \int \frac{|\rho(k')|^2\, d^3k'}{(w' - \Delta M)^2 (w' - w)}\right]} \tag{13.36}$$

This form has the same advantages as the similar equation in pair theory. In our particular version of the theory there are no divergence difficulties even for a point source, because at high energies w is proportional to k^2. Of course, the nonrelativistic energy relation no longer makes sense at very high energies. In a relativistic treatment of the mesons (Klein-Gordon equation), (13.35) is replaced by[1]

$$g_r^2 = Zg^2 = g^2 \frac{1}{1 + g^2 P \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(k)|^2}{(\omega - \Delta M)^2 2\omega}} \tag{13.37}$$

The integral in the denominator now diverges at high momenta for a point source, so that the scattering cross section would become zero for small k. Use of the renormalized coupling constant introduces an extra power of w' [or ω' if (13.37) is used] in the integral in (13.36), so that even for a point source the cross section remains finite in the limit of small momenta. In this case, however, difficulties arise which may appear even in our nonrelativistic version. In particular, from (13.35) we find

$$g^2 = \frac{g_r^2}{1 - g_r^2 P \int \frac{d^3k}{(2\pi)^3} \frac{|\rho(k)|^2}{(w - \Delta M)^2}}$$

Hence, for a sufficiently small (spatial) source and a finite (perhaps large) value of g_r, g^2 becomes negative and g imaginary. (In the relativistic theory and in the limit of a point source, $g \to -i0$.) This implies a nonhermitian Hamiltonian (13.3). Furthermore, Z, when expressed in terms of g_r, is then negative, which means a negative probability of finding a bare neutron in a physical one. To formulate such a theory[1] requires the introduction of an indefinite metric in Hilbert space, which goes beyond our framework. In a cutoff theory no trouble develops so long as the size of the source is limited to a value which reduces $Z(g_r)$ so that $0 \leq Z \leq 1$.

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[1] See, e.g., Lee, *loc. cit.* The factor (2ω) arises naturally from the use of the Klein-Gordon equation, as we saw earlier.

Formally, we find from (13.33) that the high-energy limit of the scattering cross section is given by the Born approximation with the coupling constant g. The physical reason for this and for the low-energy behavior will be discussed extensively in the next chapter.

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[1] See G. Källen and W. Pauli, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.*, **30**(7) (1955).

CHAPTER 14

Lee Model: States with $Q = -\frac{3}{2}$

14.1. Scattering: Low Equation. For the states not considered in the previous chapter the problem can also be reduced to integral equations for the wave functions of the virtual pion. In general, however, such equations cannot be solved explicitly, and hence we are not going to derive them. From an experimental point of view, the wave function of the virtual pions is actually not the quantity of primary interest, since most of the information it contains is not accessible to present observational techniques. What can be measured most easily is the scattering cross section, and therefore one should concentrate on calculating the phase shift as a function of the energy. We shall now learn a very important method for deriving general properties of the phase shift, short-cutting the calculation of the complete wave function of the mesons.

For simplicity, we restrict ourselves in this chapter to the case $\Delta M = 0$ and denote a physical nucleon state (proton or neutron) by $|N\rangle$. The two physical nucleons then have the same energy (mass), $E = 0$; the quantity we are interested in is an element of the S matrix (8.23):

$$\langle \text{out}, N + \pi_{\mathbf{k}'} \mid \text{in}, N + \pi_{\mathbf{k}} \rangle = \langle N \mid \psi^{\text{out}}(\mathbf{k}')\psi^{\dagger\text{in}}(\mathbf{k}) \mid N \rangle \tag{14.1}$$

Equations (13.9) and the similar equation for

$$\begin{aligned} \psi^{\dagger\text{in}}(\mathbf{k},t) &= \psi^{\dagger\text{out}}(\mathbf{k},t) - \frac{ig}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dt'\, e^{iw(t-t')} \rho^*(\mathbf{k}) \tau_-(t') \\ &= \psi^{\dagger\text{out}}(\mathbf{k},t) - \frac{ig}{(2\pi)^{3/2}}\, \rho^*(\mathbf{k})\, e^{iwt} \tau_-(w) \end{aligned} \tag{14.2}$$

tell us that the state with an incoming plane wave, $\psi^{\dagger\text{in}}(\mathbf{k}) \mid N\rangle$, is an outgoing plane wave with the same momentum, $\psi^{\dagger\text{out}}(\mathbf{k}) \mid N\rangle$, plus something which is generated by $\tau_{-}(w)$. Obviously this must be the scattered wave we are looking for.

If $| N\rangle$ is the proton, then we can use (13.18)—or, rather, its analogy for outgoing operators—to derive an explicit expression for $\tau_{-}(w) \,| N\rangle$. Since $\bar{\tau}^{\text{in}}_{-}(K_0)$ is proportional to $\delta(K_0)$ and since

$$\psi^{\dagger\,\text{out}}(\mathbf{k},K_0) = 2\pi\psi^{\dagger\,\text{out}}(\mathbf{k})\delta(K_0 - w)$$

we obtain

$$-\frac{ig}{(2\pi)^{\frac{3}{2}}}\,\rho^*(\mathbf{k})\bar{\tau}_{-}(w) = -\,\frac{ig^2}{\pi}\,\frac{|\rho(\mathbf{k})|^2\,km}{D_{+}(w)w}\int\frac{d\Omega}{4\pi}$$
$$\times\int_0^\infty dk'\;\delta(k - k')\bar{\psi}^{\dagger\,\text{out}}(k') = \mathscr{A}\,\psi^{\dagger\,\text{out}}(k) \qquad (14.3)$$

which shows that we have a spherically symmetric scattered wave for a spherical source. Comparison of (14.2) and (14.3) with (8.24) tells us that $\mathscr{A}$ is $\{\exp\,[2i\delta(k)] - 1\}$, which also agrees with (13.33).

In general, we can deduce the S-matrix element from (14.2) and the familiar time dependence of matrix elements between eigenstates of the Hamiltonian. Thus[1]

$$\begin{aligned}
&\langle\text{out}, N + \pi_{\mathbf{k}'}\,|\,\text{in}, N + \pi_{\mathbf{k}}\rangle \\
&\quad= \langle N\,|\,\psi^{\text{out}}(\mathbf{k}')\left[\psi^{\dagger\,\text{out}}(\mathbf{k}) - \frac{ig\rho^*(k)}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty} dt\; e^{-iwt}\tau_{-}(t)\right]|\,N\rangle \\
&\quad= \delta^3(\mathbf{k} - \mathbf{k}') - \langle\text{out}, N + \pi_{\mathbf{k}'}\,|\, - \frac{ig\rho^*(k)}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty} dt\; e^{iHt}\tau_{-}(0)e^{-iHt}e^{-iwt}\,|\,N\rangle \\
&\quad= \delta^3(\mathbf{k} - \mathbf{k}') - 2\pi i\delta(w - w')\,g\,\frac{\rho^*(k)}{(2\pi)^{\frac{3}{2}}}\langle\text{out}, N + \pi_{\mathbf{k}'}\,|\,\tau_{-}(0)\,|\,N\rangle \\
&\quad\equiv \delta^3(\mathbf{k} - \mathbf{k}') - 2\pi i\delta(w - w')T(\mathbf{k}') \qquad (14.4)
\end{aligned}$$

In obtaining (14.4), we made use of the fact that

$$\langle\text{out}, N + \pi_{\mathbf{k}'}\,|\,e^{iHt} = \langle\text{out}, N + \pi_{\mathbf{k}'}\,|\,e^{iw't}$$

since the energy of both physical nucleons was adjusted to be zero. In general, the matrix T would be a function of both $\mathbf{k}'$ and $\mathbf{k}$, but because of the spherical symmetry of the problem and energy conservation, it is here a function of a single variable $\mathbf{k}'$. The relation of the T matrix to the phase shift can be found from (14.3) or directly from (8.30) and is[2]

$$T(k) = -\,\frac{e^{i\delta(k)}\sin\delta(k)}{\pi mk}\,\frac{1}{4\pi} \qquad (14.4a)$$

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[1] It must be remembered that $\rho(\mathbf{r})$ is a spherically symmetric source, so that $\rho(\mathbf{k}) = \rho(k)$.

[2] See also, e.g., B. A. Lippman and J. Schwinger, *Phys. Rev.*, **79**:469 (1950). With our present normalization, $\underset{f}{\mathbf{S}} = \int d^3k$ and $g(E) = 1/\pi mk$.

The T matrix defined by (14.4) is, of course, time-independent. It is more convenient than the S matrix for the subsequent discussion because the singular part of the latter has been split off. Thus, the T matrix will turn out to be an analytic function of the energy.

From (14.4) it follows that the scattering amplitude $T(k)$ is obtained by analyzing $\tau_- \mid N\rangle$ into outgoing states with energy w. Using the fact that $\psi(\mathbf{k})$ commutes with τ_- and that $\psi^{\text{out}} \mid N\rangle = 0$, we can rewrite the last expression in the following form:[1]

$$\begin{aligned} T(k) &= \frac{g\rho^*(k)}{(2\pi)^{\frac{3}{2}}} \langle \text{out}, N + \pi_{\mathbf{k}} \mid \tau_- \mid N\rangle \\ &= \frac{g\rho^*(k)}{(2\pi)^{\frac{3}{2}}} \langle N \mid \left[\psi(\mathbf{k}) - \frac{ig\rho(k)}{(2\pi)^{\frac{3}{2}}} \int_0^\infty dt\, \tau_+(t)\, e^{i(w+i\epsilon)t} \right] \tau_- \mid N\rangle \\ &= \frac{ig^2 \left|\rho(k)\right|^2}{(2\pi)^3} \int_0^\infty dt\, e^{i(w+i\epsilon)t} \langle N \mid [\tau_-(0), \tau_+(t)] \mid N\rangle \end{aligned} \tag{14.5}$$

Equation (14.5) is usually called the Low equation,[2] after its progenitor.

14.2. $\pi^- + n$ Scattering. Our first application of the Low equation will be to derive some general properties of the $\pi^- + n$ scattering amplitude. The commutator in (14.5) contains a term with $\tau_+(t)$ and one with $\tau_-(0)$ in front of $\mid n\rangle$. The former generates a state with[3] $Q = \frac{1}{2}$; in fact, we see from

$$\mid \text{in}, n\rangle = Z^{\frac{1}{2}} \mid n) + \frac{U_b(\mathbf{k})\psi^\dagger(\mathbf{k})}{(2\pi)^{\frac{3}{2}}} \mid p)$$

that[4]

$$\tau_+(t) \mid \text{in}, n\rangle = Z^{\frac{1}{2}} \mid p\rangle \tag{14.6}$$

Hence this part of the right-hand side becomes[5]

$$\frac{ig^2 \left|\rho(k)\right|^2}{(2\pi)^3} \int_0^\infty dt\, e^{i(w+i\epsilon)t} \langle n \mid \tau_-(0) Z^{\frac{1}{2}} \mid p\rangle = -\frac{g_r^2 \left|\rho(k)\right|^2}{w(2\pi)^3} \tag{14.7a}$$

with

$$g_r = \langle p \mid g\tau_+ \mid n\rangle = \langle n \mid g\tau_-(0) \mid p\rangle = Z^{\frac{1}{2}} g \tag{14.7b}$$

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[1] The commutator in (14.5) emphasizes that the source must involve dynamical variables in order to scatter. For a c-number source the right-hand side of (14.5) is zero, in agreement with our previous result. The $i\epsilon$ is the usual convergence factor of Green's functions which defines the position of the pole in momentum space.

[2] F. Low, *Phys. Rev.*, **93**:1392 (1955).

[3] From $[Q, \tau_\pm(t)] = \pm\tau_\pm(t)$ we conclude that the operators τ_+ and τ_- change the eigenvalue of Q by 1.

[4] Since $\mid \text{in}, n\rangle = \mid \text{out}, n\rangle$, we shall use $\mid n\rangle$ to indicate either state.

[5] The $i\epsilon$ is not needed in the denominator of (14.7a) because the pole at $w = 0$ lies in the unphysical region.

The other term of (14.5) generates states with $Q = -\frac{3}{2}$, e.g., states with one outgoing pion and a neutron or two outgoing pions and a proton. At this point we have to assume that the form of the energy spectrum is not changed by the interaction and that there is no discrete state with $Q = \frac{3}{2}$. In that case these states have energies $>m$. If we denote the physical states with $Q = -\frac{3}{2}$ by $|\,\text{out}, i\rangle$, we can write this part of the commutator as

$$\begin{aligned}\langle n \,|\, \tau_+(t)\tau_-(0) \,|\, n\rangle &= \sum_i \langle n \,|\, e^{iHt}\tau_+(0)e^{-iHt} \,|\, \text{out}, i\rangle\langle \text{out}, i \,|\, \tau_-(0) \,|\, n\rangle \\ &= \sum_i e^{-iE_it} \,|\, \langle \text{out}, i \,|\, \tau_- \,|\, n\rangle \,|^2 \qquad (14.8)\end{aligned}$$

where E_i is the energy of the state $|\,\text{out}, i\rangle$. This gives us, all together,

$$T(k) = -\frac{g_r^2\,|\rho(k)|^2}{(2\pi)^3 w} - \frac{|\rho(k)|^2}{(2\pi)^3}\sum_i \frac{|\,\langle \text{out}, i \,|\, \tau_- g \,|\, n\rangle\,|^2}{E_i - w - i\epsilon} \qquad (14.9)$$

A first remark concerns the sign of $\lim_{k\to 0} \delta(k)/k$. It appears from (14.9) and (14.4*a*) that it is >0, as for an attractive interaction without a bound state. The reason why the π^- interaction with a neutron is attractive can be traced back to be the following. A single neutron emits a pion, and the proton so formed absorbs it. We found that these processes lower the energy by $\mathscr{E}_0$. If another π^- comes along, then the emission and absorption activities of the nucleons become more violent. We have seen that a source emits more eagerly if an identical boson is around, because of Einstein's well-known induced emission.[1] This process decreases the energy, the gain of interaction energy outweighing the pion rest energy. Thus the interaction will decrease the energy of a $n + \pi^-$ below the energy of an $n + \pi^-$ without interaction, imitating an attractive force. That such a situation gives an attraction, whereas for a $p + \pi^-$ the phase shift starts negative, as we saw in the previous chapter, is the crux of our present understanding of low-energy pion physics.

The statements made so far are exact. However, (14.9) cannot be solved in closed form. We shall find its solution in the approximation that, in $\sum_i$ over $Q = -\frac{3}{2}$ states, those corresponding to $|\,\text{out}, p + 2\pi^-\rangle$ are neglected. If these virtual production processes of a second pion are

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[1] Similarly, we can interpret the Yukawa potential as arising from the emission of mesons by one nucleon induced by the meson field of the other. The stimulated-emission probability arises formally because

$$|\langle n + 1 \,|\, \phi \,|\, n\rangle|^2 \propto (n + 1)$$

left out,[1] then the term in (14.9) that is proportional to $\sum_i$ involves only $\langle \text{out}, n + \pi^- \mid \tau_- \mid n \rangle$, which can be expressed by T. We find[2,3]

$$T(k) = T(k) = -\frac{g_r^2\,|\rho(k)|^2}{w(2\pi)^3} - |\rho(k)|^2 \int d^3k' \,\frac{|T(k')|^2/|\rho(k')|^2}{w' - w - i\epsilon} \tag{14.10}$$

Although (14.10) is a nonlinear integral equation, it can be solved exactly. To this end we recognize that $T(k)/|\rho(k)|^2$ is the boundary value of a function which is analytic in the complex w plane. Indeed, (14.10) states that the function $\mathfrak{H}$ of a complex variable z which obeys

$$\frac{1}{\mathfrak{H}(z)} = \frac{g_r^2}{z} + \int_m^\infty \frac{dw'\,|\rho(k')|^2\,k'm}{2\pi^2\,|\mathfrak{H}(w' + i\epsilon)|^2\,(w' - z)} \tag{14.11}$$

is related to T by

$$T(k) = -\lim_{\epsilon\to 0} \frac{|\rho(k)|^2}{(2\pi)^3}\,\frac{1}{\mathfrak{H}(w + i\epsilon)} \tag{14.12}$$

From (14.11) we see that $1/\mathfrak{H}(z)$ is analytic in the complex plane except for a pole at the origin and a cut on the real axis from m to ∞. At this cut the imaginary part of $\mathfrak{H}$ is discontinuous:

$$\begin{aligned}\lim_{\epsilon\to 0}&\left[\frac{1}{\mathfrak{H}(w + i\epsilon)} - \frac{1}{\mathfrak{H}(w - i\epsilon)}\right]\\ &= \int_m^\infty \frac{dw'}{2\pi^2}\,\frac{|\rho(k')|^2\,k'm}{|\mathfrak{H}(w' + i\epsilon)|^2}\left(\frac{1}{w' - w - i\epsilon} - \frac{1}{w' - w + i\epsilon}\right)\\ &= \frac{i\,|\rho(k)|^2\,mk}{\pi\,|\mathfrak{H}(w + i\epsilon)|^2}\end{aligned}$$

[1] This approximation will also be used in pion physics. In general, such an approximation will not be good, especially at high energies. Indeed, comparing the exact high-energy limit (14.17) with the one of this solution, Eq. (14.14), we see that they may be completely different if $Z \ll 1$. However, the solution may have a degree of validity at low energies, where it approaches the correct limit.

[2] By similar techniques, we obtain for $p + \pi^-$ scattering the exact equation

$$T(k) = \frac{g_r^2\,|\rho(k)|^2}{w(2\pi)^3} - |\rho(k)|^2 \int d^3k' \,\frac{|T(k')|^2}{|\rho(k')|^2\,(w' - w - i\epsilon)}$$

the precise solution of which was obtained in the last chapter:

$$T(k) = \frac{g^2}{8\pi^3}\,\frac{|\rho(k)|^2}{wD_+(w)} = \frac{g_r^2\,|\rho(k)|^2}{8\pi^3 w\left[1 + g_r^2 w \displaystyle\int \frac{d^3k'}{8\pi^3}\,\frac{|\rho(k')|^2}{w'^2(w' - w - i\epsilon)}\right]} \tag{14.5a}$$

[3] Because of our normalization of the one-meson states

$$\langle \text{in}, \mathbf{k} \mid \text{in}, \mathbf{k}' \rangle = \langle N \mid \psi^{\dagger\text{in}}(\mathbf{k})\,\psi^{\text{in}}(\mathbf{k}') \mid N \rangle = \delta^3(\mathbf{k} - \mathbf{k}')$$

the sum $\sum_i$ of (14.9) becomes $\int d^3k$ in the continuum limit.

Since $\mathfrak{H}(w - i\epsilon) = \mathfrak{H}^*(w + i\epsilon)$, the discontinuity of $\mathfrak{H}$ is directly related to ρ:

$$-i \lim_{\epsilon \to 0} [\mathfrak{H}(w + i\epsilon) - \mathfrak{H}(w - i\epsilon)] = -\frac{|\rho(k)|^2\, mk}{\pi} = 2\,\mathrm{Im}\,\mathfrak{H}(w + i\epsilon)$$

A function with this kind of analytic behavior is

$$\mathfrak{H}(w) = \frac{w}{g_r^2}\left[1 - wg_r^2 \int_m^\infty \frac{dw'\,|\rho(k')|^2\, k'm}{2\pi^2 w'^2(w' - w - i\epsilon)}\right] \tag{14.13}$$

provided that g_r is not so large that the bracket has a zero for $w < m$. With this proviso, (14.11) is actually satisfied by (14.13).[1]

To relate this solution to more familiar material, we shall also obtain it by means of methods used in pair theory. The integral in (14.11) can be written as a contour integral around the cut (see Fig. 13.2) and, furthermore, can be closed by an infinite circle, since $\mathfrak{H}(z) \to \infty$ for $z \to \infty$. Hence, (14.11) can be written in the linear form

$$\frac{1}{\mathfrak{H}(z)} = \frac{g_r^2}{z} + \frac{1}{2\pi i}\int_C \frac{dz'}{\mathfrak{H}(z')(z' - z)}$$

where C is the path of integration shown in Fig. 13.2. Evaluating the integral with Cauchy's theorem, we see immediately that (14.11) is satisfied. Thus we arrive at the following solution for $T(k)$ [compare this with (14.12) and (14.13)]:

$$T(k) = \frac{-g_r^2\,|\rho(k)|^2}{(2\pi)^3 w\left[1 - wg_r^2 \displaystyle\int \frac{d^3k'}{(2\pi)^3}\frac{|\rho(k')|^2}{(w' - w - i\epsilon)w'^2}\right]} \tag{14.14}$$

Remembering that $T = -\sin(\delta e^{i\delta}/4\pi^2 km)$, we get for the phase shift

$$\tan\delta(k) = \frac{g_r^2\,|\rho(k)|^2\, km}{2\pi w\left[1 - g_r^2 wP \displaystyle\int \frac{d^3k'}{(2\pi)^3}\frac{|\rho(k')|^2}{(w' - w)w'^2}\right]} \tag{14.15}$$

This is of the same form as the phase shift in pair theory with attraction [see (12.11*b*)]. The forms of (14.14) and (14.15) point out one advantage of the Low equation: because it involves only physical states, it allows the solution to be expressed directly in terms of the physically meaningful, or "measurable," coupling constant g_r, rather than g. We assumed that there is no resonance at negative energies, which would indicate discrete states with $Q = -\frac{3}{2}$.¶ This imposes a limit on g_r^2, which depends on $\rho(k)$. For sufficiently large g_r, within this limit, a resonance for $w > m$ will emerge. In any case, the denominator enhances the phase shift, whereas it was damped for the $p + \pi^-$ scattering. Since for low energies the inelastic terms in (14.5) leading to a $p + 2\pi^-$ state act in the same direction as those for the $n + \pi^-$ system, it is to be expected that the exact solution will also show this feature. Therefore we anticipate that at low or medium energies the $n + \pi^-$ cross section will become much larger than the one for $p + \pi^-$ scattering. This can also be seen by comparing (14.14) with (14.5*a*). The latter solution for $p + \pi^-$ scattering can also be obtained by the technique we used to derive (14.14).

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[1] For a discussion of the uniqueness of the solution, see L. Castillejo, R. H. Dalitz, and F. J. Dyson, *Phys. Rev.*, **101**:453 (1956); and M. Ida, *Progr. Theoret. Phys. (Kyoto)*, **21**:625 (1959).

14.3. Low- and High-energy Behavior of $T(k)$. Finally, we shall derive from (14.9) and (14.5*a*) the low- and high-energy behavior of $T(k)$. To this end we consider

$$t(k) = -\frac{(2\pi)^3 T(k)}{|\rho(k)|^2}$$

as a function of w. From the factors in the denominators we see that the term $\sum_i$ in (14.9) is regular for $w < m$, since $E_i > m$. However, the first term has a pole at $w = 0$ with a residue g_r^2. Therefore, this term will be dominant for $w \to 0$, and we have

$$\lim_{w\to 0} t(k) = \frac{g_r^2}{w} \tag{14.16}$$

Hence we find the same result for the $n + \pi^-$ system as for the $p + \pi^-$ scattering [see (14.5*a*)], namely, that for $w \to 0$ the exact result is the Born approximation calculated with the renormalized coupling constant.[1] In the limit $w \to \infty$, if the sum converges sufficiently rapidly, we may put the denominator under $\sum_i$ in (14.9) equal to $-w$. Taking the factor $1/w$ out of the whole expression, we see that what is left is just the equal-time commutator. This, however, is $-\tau_3$, so that we get

$$\lim_{w\to\infty} t(k) = \frac{g^2}{w}\langle n \mid -\tau_3 \mid n\rangle = \frac{g^2}{w}(2Z - 1) \tag{14.17}$$

The last expression becomes more suggestive if we write it as

$$\frac{g^2}{w}Z + \left(-\frac{g^2}{w}\right)(1 - Z)$$

¶ In that case, (14.14) would be the solution of a Low equation in which the contribution of this discrete state is included.

[1] It is always assumed that $|\rho(k)|^2$ can be continued analytically to the unphysical energy $w = 0$ or $k^2 = -2m^2$.

which is the [Born approximation of $t(k)$ for $n + \pi^-$] × (probability of finding a bare n in $|n\rangle$) + [Born approximation of $t(k)$ for $p + \pi^-$] × (probability of finding a bare p in $|n\rangle$). In this form the result is analogous to the one for the $p + \pi^-$ scattering. In fact, with the Low equation we could have derived the latter without having to solve the Schrödinger equation.

The physical significance of the low- and high-energy limits can be seen by resolving the scattering process into elementary interactions between bare particles. This corresponds to an expansion of T in powers of g. The Born approximation consists in taking the least number of processes. For the scattering, it corresponds to an absorption of the original π^- and an emission of the scattered pion. For the $n + \pi^-$ scattering, the processes must occur in reverse order from that for $p + \pi^-$ processes, as is illustrated by the graphs[1] of Fig. 14.2. Higher orders in g give a rescattering of the pion and a dressing of the neutron. For the order g^4 this is illustrated in Fig. 14.3. Now, imagine that the energy of the external pion[2] is increased beyond that of all virtual pions. In this case the time between emission and absorption, ΔT, has to be as short as possible since the uncertainty in the nucleon's energy, ΔE, is as large as the energy of the external pion (which it absorbs and emits), and $\Delta E\,\Delta T \sim w\,\Delta T \sim 1$. Hence, for $p + \pi^-$ scattering, the contribution from the graph of Fig. 14.4a will dominate

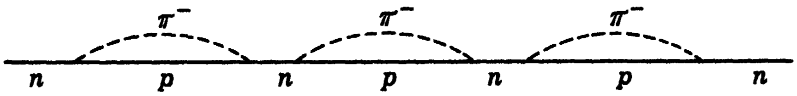

Fig. 14.1. Graph for the physical neutron.

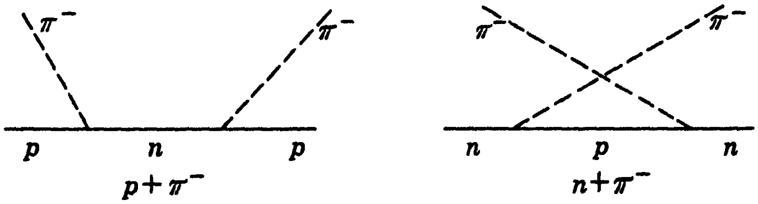

Fig. 14.2. Diagrams for the scattering of π^- on protons and neutrons.

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[1] In these diagrams the lines show the paths of the various bare particles, and time flows from left to right. In such a picture the physical neutron looks as shown in Fig. 14.1. These diagrams should not be taken too literally, since the concept of a classical path does not apply to virtual particles. However, they serve to illustrate the various terms in a perturbation expansion where H' effects the elementary emission (and absorption) process between bare particles. See G. C. Wick, *Revs. Modern Phys.*, **27**:339 (1955).

[2] This is the one which is scattered.

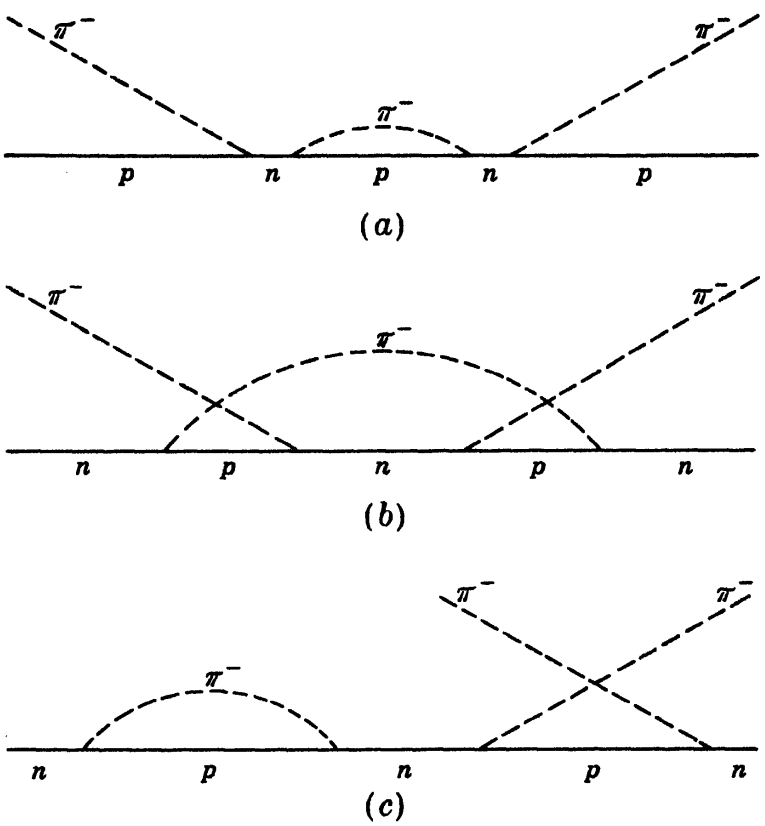

Fig. 14.3. Graphs of order g^4 for (*a*) $p + \pi^-$ scattering and (*b*, *c*) $n + \pi^-$ scattering.

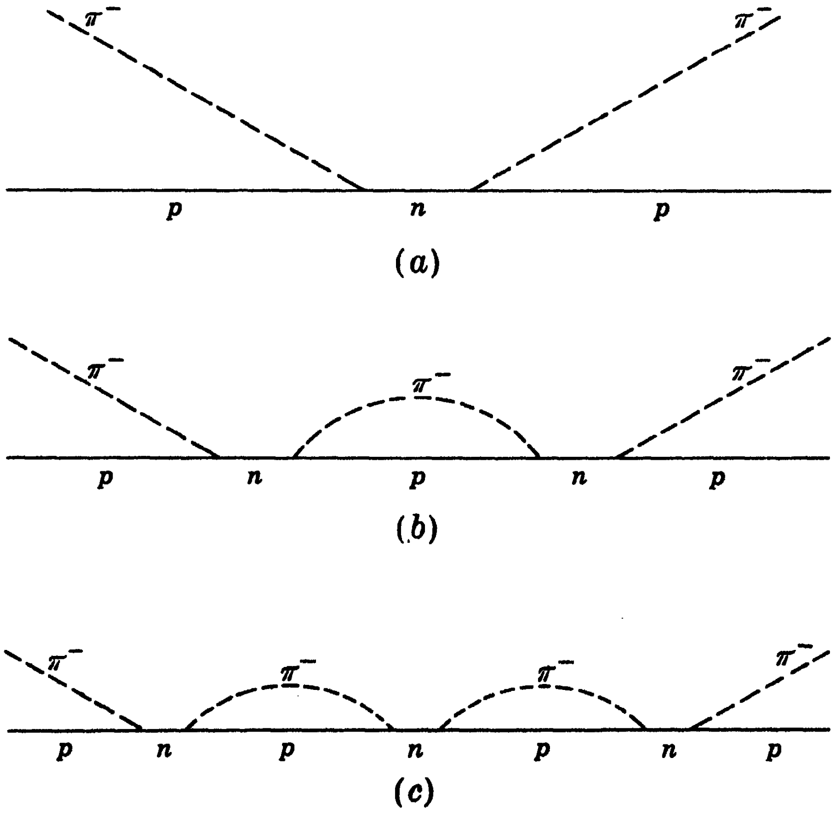

Fig. 14.4. Diagrams to illustrate the high-energy behavior of $p + \pi^-$ scattering.

over all those diagrams in which (intermediate) virtual pions are present, such as those of Fig. 14.4*b* and *c*. The high-energy limit for $p + \pi^-$ scattering is therefore given by the Born approximation. For the $n + \pi^-$ system the scattering can also occur during the time that it is a bare proton. Thus, in the high-energy limit we get here both scattering by a bare neutron, as in Fig. 14.5*a*, and scattering by a bare proton, as in Fig. 14.5*b*, but in both cases without (intermediate) virtual mesons being emitted and absorbed between the interaction with the external meson. The latter contributions, illustrated in Fig. 14.5*c*, will again be small compared with the others, and we get the Born-approximation amplitudes from the two kinds of particles weighted by the corresponding probabilities.

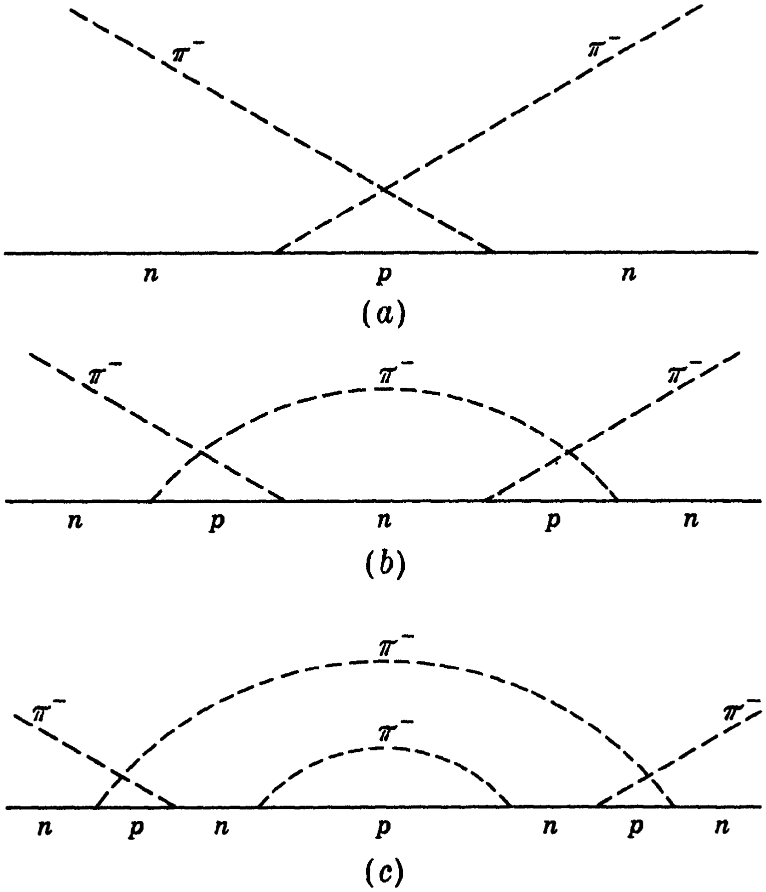

Fig. 14.5. Graphs to illustrate the high-energy behavior of $n + \pi^-$ scattering.

To discuss the low-energy limits, let us consider first the fictitious situation in which the external meson has no rest energy, so that its energy can be made much smaller than that of the virtual pions. In this circumstance it follows from the uncertainty argument that the time between the interactions with the external particle will last much longer than the virtual processes. For $p + \pi^-$ scattering, this means that, after absorbing the pion, the nucleon practically becomes a physical neutron, as shown in Fig. 14.6. But the absorption of the external pion converts the proton into a bare neutron and, therefore, can happen only during the fraction of time that the physical neutron is a bare neutron. Since two such elementary processes are involved, we obtain the following expression for the cross section in this limit: (Born approximation) × (probability of having a bare neutron in the physical neutron)^2. The same holds for $n + \pi^-$ scattering, where only higher-order corrections of the form illustrated in Fig. 14.7*a*, but not those of the form shown in Fig. 14.7*b*, contribute in this limit. For $m \neq 0$ the above conclusions still hold in the limit $w \rightarrow 0$, but this is in an unphysical region, and the statements about the low-energy limits are true only in an extrapolated sense. Generally speaking, we can say from the way that g is introduced that it measures the strength of the elementary interaction between the bare nucleon and pion. On the other hand, g_r gives the strength of the elementary interaction between the physical nucleon and pion,[1] as appears from $g_r = \langle p \mid \tau_+ g \mid n \rangle$. Fast processes involve the bare source particles and hence g; slow ones

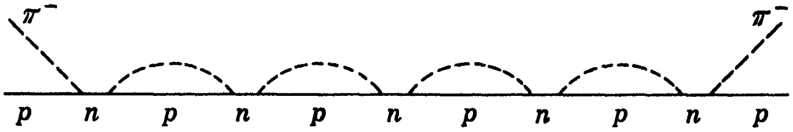

Fig. 14.6. Diagram to illustrate the "low-energy limit" of $p + \pi^-$ scattering.

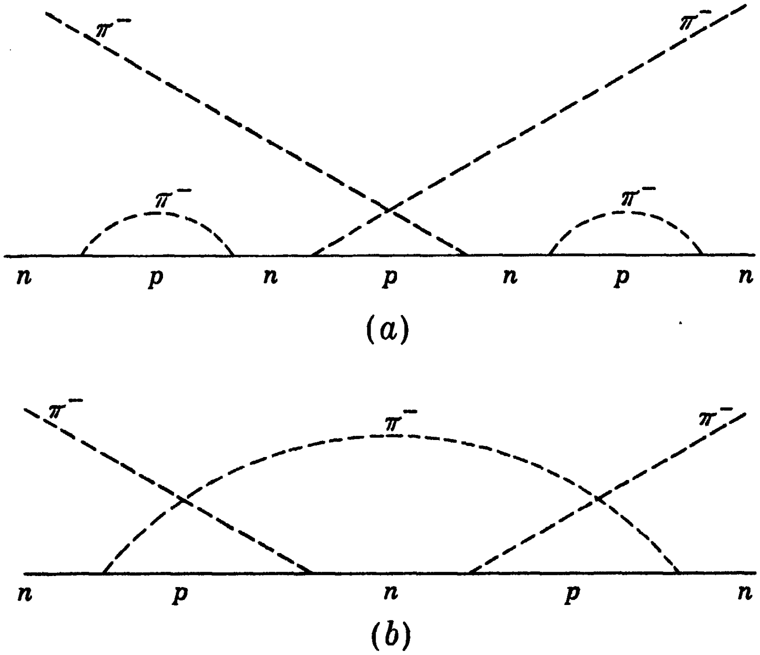

Fig. 14.7. Graphs for the discussion of the "low-energy limit" of $n + \pi^-$ scattering.

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[1] The physical and bare pions are identical.

involve the physical source particles and g_r. Since at high energies the situation becomes very complex,[1] it is generally only possible to measure g_r. Even this quantity cannot be obtained directly, since the energy $w = 0$ is not available experimentally. However, if the low-energy-scattering phase shifts can be extrapolated to this energy, or if the bracketed term in the denominators of (14.14) and (14.5*a*) is approximated by an expansion in terms of g_r for small, but physical, energies, then we can obtain g_r^2 directly from measured cross sections. We shall discuss this further in the last part of the book.

Various modifications and generalizations of the Lee model have been studied from time to time. One of these, proposed by van Hove, still limits the number of degrees of freedom but allows more than one meson in the cloud of the physical nucleons. The generalization consists in allowing only one type of field quantum, π^0, but two processes characterized by

$$V \rightleftharpoons n + \pi^\circ \qquad \text{(coupling strength } g_a\text{)}$$
$$n \rightleftharpoons V + \pi^\circ \qquad \text{(coupling strength } g_b\text{)}$$

The theory has several advantages over the Lee model and is also exactly soluble for the bound physical states. The main advantage for us is that it shows the connection between the neutral scalar theory of Chaps. 9 and 10 and the Lee model. Thus, if the physical masses of the V and n are the same and $g_a = g_b$, then the theory reduces to the neutral scalar theory, except for a doubling of states. To discuss the theory in detail would lead us too far astray, and we refer the reader to the original articles.[2]

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[1] It is clear that the nonrelativistic approximation cannot be made at such energies; furthermore, nucleon-pair creation and recoil must be taken into account.

[2] T. W. Ruijgrok and L. van Hove, *Physica*, **22**:880 (1956); L. van Hove, *Physica*, **25**:365 (1959); T. W. Ruijgrok, *Physica*, **24**:185, 205 (1958) and **25**:357 (1959).

Part Three

PION PHYSICS

CHAPTER 15

Introduction

15.1. The Static Model. There are many problems in physics which fall within the scope of quantum field theory and can be treated by methods similar to those we have developed or by less rigorous ones. The prototypes of discrete coupled oscillators are the atoms in a crystal. Electrons that pass through the crystal act as a disturbance that is linearly coupled to the atoms. However, the reaction of the crystal back on the electrons cannot be neglected, so that the interaction is a dynamic one, involving the translational degrees of freedom of the electron, and cannot be represented as a coupling to a fixed and prescribed source. The ensuing complications make this "polaron" problem[1] an interesting one, but one that cannot be solved exactly.

An example of a mechanical model with a quasi continuum is the quantization of the equations of motion of a liquid.[2] The complications of this problem arise because of the nonlinearity of the hydrodynamical equations, so that only approximate solutions exist, e.g., at low temperatures, where quantum effects dominate the picture and the continuum approach appears to give reasonable results.[3] In nuclear physics the surface waves of nuclei represent a two-dimensional example in which quantum effects are important.[4] Also, the many-body problem (such as represented by a Bose gas or a Fermi-Dirac gas of electrons or nucleons), which at first sight appears to have no field features, is most easily tractable by methods of field theory.[1] Furthermore, in the high-density limit an electron gas with Coulomb interactions turns out to be equivalent to a pair theory and can therefore be solved exactly by the methods we have described.[2]

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[1] T. D. Lee, F. E. Low, and D. Pines, *Phys. Rev.*, **90**:297 (1953); and T. D. Lee and D. Pines, *Phys. Rev.*, **92**:883 (1953).

[2] R. Krönig and A. Thellung, *Physica*, **18**:749 (1952).

[3] L. D. Landau and E. M. Lifschitz, "Statistical Physics," chap. 6, Addison-Wesley Publishing Company, Reading, Mass., 1958.

[4] A. Bohr and B. R. Mottelson, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.*, **27**:16 (1953).

The most fascinating applications of our rules are, however, not to any material substance but to immaterial fields, the excitations of which appear to us as elementary particles. There are many examples—among them, quantum electrodynamics and the theory of "weak interactions." These theories have the advantage that the coupling is sufficiently small that the physical particles and bare particles are almost identical. On the other hand, their systematic development requires the theory of the representations of the Lorentz group and is therefore outside the scope of this book. Instead, we shall be concerned with an approximate theory of the pion-nucleon interaction. This theory is one of the most exciting ones, inasmuch as it penetrates to the smallest distances yet explored by any partially successful physical theory. In the other applications we have named, the phenomena take place in regions of atomic dimensions, or of the order of the Compton wavelength of the electron. By now, we can be fairly confident that the principles we have employed are valid in the latter regions. Pion physics, however, occurs at distances of the order of 10^{-13} cm, and it is interesting to note that the concepts of field theory seem to work at this small a distance. Even though much work has been done, very little is yet known about fields inside this region, nor is it known whether it is necessary to introduce the concept of a smallest distance. These problems are among the most challenging that face physicists today.

The theory of the pion-nucleon interaction is complicated by two facts: (1) the exact form of the interaction is not known for certain, and (2) particles other than pions make up the structure of the physical nucleon. It is fairly certain, for example, that the meson cloud in the nucleon also contains *K* mesons and hyperons, as well as antinucleons and antihyperons. An approximate separation of part of the pion cloud from that of the other particles fortunately is possible, because the pions extend to the largest spatial dimension. As we saw earlier, the extension of the cloud is roughly measured by its Compton wavelength; in other words, it is inversely proportional to the particle mass that the field represents. The *K* meson is approximately $3\frac{1}{2}$ times heavier than the pion, and because our present concepts indicate that "strangeness" is conserved, its emission requires changing the nucleon into a hyperon, which costs approximately another meson mass. Hence the cloud of K mesons and hyperons is not expected to extend to more than about two-ninths the distance of the pions, which implies that the volume of the nucleon is 99 per cent pion cloud. Nucleon-antinucleon pairs have an even shorter range. However, their importance may be that a pion can split into three pions[1] through an intermediate pair, thus contributing to a finite size of the pion itself.[2]

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[1] See, e.g., K. Huang and C. N. Yang, *Phys. Rev.*, **105**:767 (1957); and J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.*, **108**:1175 (1957).

[2] G. Wentzel, *Phys. Rev.*, **108**:1593 (1957).

These considerations show that the prospects for a theory containing only pions and a nucleon are fairly good. Experimental evidence[3] indicates that the nucleon has a "repulsive core," which is not understood and which extends to about 5×10^{-14} cm, but that outside this region pions are the principal contributors to the structure. The situation is not as clear-cut as, say, the nonrelativistic theory of the hydrogen atom. The corrections to this theory are important in a region of the order of the Compton wavelength of the electron, or $\frac{1}{137}$ of the radius of the hydrogen atom. Correspondingly, the Balmer formula is a very good approximation, and fine-structure corrections are less than 0.1 per cent. Here, perhaps an accuracy of 5 per cent is more appropriate.

Even with the restriction to pions and nucleons, the picture is complicated. The reason is that in strongly interacting systems many virtual particles are present and all sorts of interactions that are allowed by invariance principles will take place. For instance, there is no reason why there should be no strong pion-pion interaction. This will certainly also affect the pion-nucleon interaction, since the nucleon is surrounded by a fairly dense pion cloud. Fortunately, it turns out that in low-energy pion physics there are large effects, medium-sized effects, and small effects. The large effects have a common origin, namely, an unstable excited state of the nucleon. The latter can be obtained by the interaction term we shall use. The medium-sized effects amount to about 15 per cent corrections to the large effects and are due to pion-pion interactions, nonlinear pion-nucleon interactions, kinematical corrections, etc. Since these are missing in our model, we shall not be able to make predictions about effects which are not directly related to the excited state. Techniques for calculating medium-sized effects are now being developed. These methods are generalizations of the ones we shall use for our model. Finally, there are small effects ($\sim$few per cent), which may be due to strange-particle interactions, many-meson forces, etc., and which are beyond the reach of our present calculational power.

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[1] Two pions are not allowed by angular momentum and parity conservation.

[2] A suggestion such as this was advanced by I. E. Tamm, *J. Exptl. Theoret. Phys.* (*U.S.S.R.*), **32**:178 (1957), trans. in *Soviet Phys. JETP* (*U.S.S.R.*), **5**:41 (1957). This would also contribute to a pion-pion interaction, for which there are some experimental indications at high energies. [See, e.g., L. S. Rodberg, *Phys. Rev. Letters*, **3**:58 (1959); W. R. Frazer and J. R. Fulco, *Phys. Rev. Letters*, **2**:365 (1959).] Quantitative results are only just being developed, and we shall therefore not consider this matter here.

[3] See, e.g., H. A. Bethe and P. Morrison, "Elementary Nuclear Theory," 2d ed., pp. 130, 132, John Wiley & Sons, Inc., New York, 1956.

With this orientation, we turn to the question of the detailed form of the pion-nucleon interaction. The empirically known processes $p \leftrightarrow p + \pi^0$, $p \leftrightarrow n + \pi^+$, $n \leftrightarrow p + \pi^-$ show (1) that there must be some linear coupling, since quadratic couplings, for example (which cannot be ruled out), would create meson pairs, and (2) that the coupling involves charge degrees of freedom. We saw in the Lee model that the latter condition allows scattering to occur even for a fixed source, that is, neglecting the translational degrees of freedom of the nucleon. Of course, momentum conservation requires that the latter be involved in the elementary process, but theory and experiments indicate that for many physical phenomena recoil effects are smaller than those due to the changes in the charge degrees of freedom. This can be seen as follows. The process of emission and reabsorption of a virtual pion with momentum $\mathbf{k}$ lasts for a time of the order of $\Delta t \sim \omega^{-1} = (\mu^2 + k^2)^{-\frac{1}{2}}$, where μ is the mass of the pion. The recoil velocity of the nucleon of mass M is $v = k/M$, so that the fluctuations in position of the nucleon, owing to the virtual processes, are of the order of

$$\Delta r \sim v\,\Delta t \sim \frac{k}{M}(k^2 + \mu^2)^{-\frac{1}{2}}$$

This uncertainty is thus less than the Compton wavelength of the nucleon, $1/M \approx 2 \times 10^{-14}$ cm. Correspondingly, the scattering cross section due to Galilean invariance and other recoil effects[1] is of the order of $(1/M)^2$ and vanishes for $M \to \infty$. On the other hand, the characteristic length for the emission and absorption process is the Compton wavelength of the meson [actually $(\mu^2 + k^2)^{-\frac{1}{2}}$], which is approximately seven times that of the nucleon. This length also determines the size of the scattering cross section due to the internal degrees of freedom. Thus, we expect to be on fairly safe ground in neglecting recoil effects for meson kinetic energies of, say, $\leqslant 3\mu$.

In the limit of $M \to \infty$ there are only two angular-momentum states accessible to the meson in the elementary process $N \to \pi + N$. Since a nucleon at rest has no orbital angular momentum and spin $\frac{1}{2}$, the conservation of total angular momentum dictates that the pion can only be emitted with orbital angular momentum 0 or 1. Since these two states have opposite parity, emission can only occur into one *or* the other state if parity is conserved; which occurs depends on the intrinsic parity[1] of the pion. Experimental evidence[2] has definitely established that this parity is negative, so that the pions must be emitted into a state of orbital angular momentum $l = 1$. This simple model is, in fact, able to account for almost all essential empirical facts. The emission of the pion may be accompanied by a flip of the nucleonic spin, so that it is necessary to take into account the two spin states of the nucleon. This is done by simply introducing the set of Pauli matrices[3] $\boldsymbol{\sigma}$, which represent the spin pseudovector of the source in the 2 × 2 spin space. In the usual representation we have

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[1] E. M. Henley and M. A. Ruderman, *Phys. Rev.*, **90**:719 (1953).

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Another property of the pion-nucleon interaction is that the invariance of the three-dimensional orthogonal group, which we considered in Chap. 7, is not destroyed in the presence of the source.[4] This has been amply demonstrated experimentally[5] in both pion and nuclear physics and will be used by us as a guiding principle. Since the charge space of the pions must be coupled to that of the nucleons in order to conserve the total charge (just as in the Lee model), it is essential to introduce the notion of a nucleon with two charge states represented by the proton and the neutron. In this new 2 × 2 space for the source, we introduce another set of Pauli matrices, $\boldsymbol{\tau}$, the isospin of the nucleon, the properties of which are already familiar to us from the Lee model.

The most general coupling between mesons and nucleons that is linear and has the properties described above can be formulated in a simple manner, taking into account the invariance properties of the Lagrangian. The odd intrinsic parity of the pion implies that the pion field changes sign under a reflection of the coordinates in the origin. Since L' is to contain ϕ linearly, it must be coupled to the source in such a manner as to form a true scalar; that is, L' must be even under reflections in the origin. This dictates a coupling of the form[1] $\boldsymbol{\sigma} \cdot \nabla\phi$. If we then have a spherical source $\rho(r)$ to ensure angular-momentum conservation, we can write

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[1] This was defined in Chap. 5.

[2] H. A. Bethe and F. de Hoffmann, "Mesons and Fields," vol. II, sec. 28*c* and *d*, Row, Peterson & Company, Evanston, Ill., 1955.

[3] The properties of the spin matrices are the same as in elementary quantum mechanics; see L. I. Schiff, "Quantum Mechanics," 2d ed., sec. 33, McGraw-Hill Book Company, Inc., New York, 1955.

[4] This statement is not exact; it holds only if electromagnetic forces, for example, are neglected.

[5] See, e.g., Bethe and de Hoffmann, "Mesons and Fields," secs. 30, 31; and Bethe and Morrison, *op. cit.*, pp. 97, 115, 128.

$$L' = \int d^3r \; \rho(r)\boldsymbol{\sigma} \cdot \nabla\phi(r) \tag{15.1}$$

As is required, this is invariant under a combined rotation of $\boldsymbol{\sigma}$ and $\mathbf{r}$, and it commutes with $\mathscr{P}_-$ (see Sec. 5.2). In the next section we shall construct the angular-momentum eigenfunctions of the pion-nucleon system from this Lagrangian and shall see explicitly that the ∇ operator causes emissions of pions only in states of orbital angular momentum unity.

It remains to introduce "isospin conservation," which implies invariance under rotations in charge space (see Chap. 7). There are three charge states of the pion, π^+, π^0, π^-. Since $\phi_\alpha(\alpha = 1, 2, 3)$ transforms like the components of a vector in charge space and since the only other vector we have at our disposal is $\boldsymbol{\tau}$, we finally obtain for the static coupling[2]

$$L' = -\frac{f}{\mu}\sum_\alpha \int d^3r \; \rho(r)\tau_\alpha\boldsymbol{\sigma} \cdot \nabla\phi_\alpha(\mathbf{r}) \tag{15.2}$$

From our earlier physical discussion, we expect the source $\rho(r)$ to contain the effects not specifically included in (15.2), such as K mesons and some vestige of recoil. It is therefore expected to have some structure with a range of perhaps $\frac{1}{4}$ to $\frac{1}{5}\mu$. We shall assume that the source is normalized to unity, $\int\rho(r)\,d^3r = 1$, which should be kept in mind when we state that the coupling constant f measures the strength of the pion-nucleon interaction. This coupling constant is determined from experiments which we shall discuss in subsequent chapters.

Of the various applications of the theory of pion-nucleon interaction, the most successful is pion-nucleon scattering. We shall also discuss electromagnetic phenomena which are qualitatively accounted for by the theory. Finally, we shall turn to the oldest historical application, namely, nuclear forces. These can be explained to a large extent by this theory.

15.2. Commutation Relations and Equations of Motion. Having decided on the interaction term, we are now in a position to write the fundamental equations of our theory. We shall do this in terms of the various representations for the field variables which we introduced in Chap. 5:[1]

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[1] The spin $\boldsymbol{\sigma}$, like the angular-momentum operator $\mathbf{L}$, is an axial vector and does not change sign under reflection. It is related to the spin of the nucleon, $\mathbf{s}$, by $\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$.

[2] The pion mass μ is introduced in this equation, so that f becomes a dimensionless coupling constant. To distinguish between $\mathbf{r}$ space and isospace, we shall henceforth use English-letter subscripts for the former and Greek-letter subscripts for the latter.

$$\phi_\alpha(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{a_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}} + a_\alpha^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\omega)^{\frac{1}{2}}}$$

$$= \int_0^\infty \frac{dk}{(2\omega)^{\frac{1}{2}}} \sum_{l,m} U_k^l(r)[a_{\alpha lm}(k)Y_l^m(\vartheta,\varphi) + a_{\alpha lm}^\dagger(k)Y_l^{*m}(\vartheta,\varphi)]$$

The two expansions emphasize different aspects of the problem, and each of them will be used at some time in the future chapters. In order to expand

$$\boldsymbol{\sigma} \cdot \nabla\rho(r) = \frac{\rho'(r)\boldsymbol{\sigma}\cdot\mathbf{r}}{r}$$

into spherical harmonics, it is useful to introduce circular, rather than linear, spin components,

$$\boldsymbol{\sigma} \cdot \frac{\mathbf{r}}{r} = (2)^{-\frac{1}{2}}(\sigma_x - i\sigma_y)(2)^{-\frac{1}{2}}\frac{x + iy}{r} + (2)^{-\frac{1}{2}}(\sigma_x + i\sigma_y)(2)^{-\frac{1}{2}}\frac{x - iy}{r} + \sigma_z\frac{z}{r}$$

$$= \sum_{m=-1}^{1} \left(\frac{4\pi}{3}\right)^{\frac{1}{2}} \sigma^m Y_1^{*m} \tag{15.3}$$

$$\sigma^{-1} = (2)^{\frac{1}{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma^{+1} = (2)^{\frac{1}{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and similarly for the isospin,

$$\sum_{\alpha=1}^{3} \tau_\alpha\phi_\alpha = (2)^{-\frac{1}{2}}(\tau_1 - i\tau_2)(2)^{-\frac{1}{2}}(\phi_1 + i\phi_2)$$

$$+ (2)^{-\frac{1}{2}}(\tau_1 + i\tau_2)(2)^{-\frac{1}{2}}(\phi_1 - i\phi_2) + \tau_3\phi_3 = \sum_{\alpha=-1}^{1} \tau^\alpha\phi^{*\alpha} \tag{15.4}$$

where[2]

$$\phi^{\pm 1} = (2)^{-\frac{1}{2}}(\phi_1 \pm i\phi_2) \equiv \phi_\pm$$

are the operators diagonalizing $t^{(3)}$. To distinguish between linear and circular components, we shall use subscripts for the former (going from 1 to 3) and superscripts for the latter (assuming the values -1, 0, 1).

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[1] In order to satisfy the canonical commutation rules the operators a are defined such that $\phi \propto -i(a - a^\dagger)$.

[2] The operators $\phi^{\pm 1}$ were written as $\phi_\pm$ in Chap. 7, where the operator $t^{(3)}$ is defined.

With these conventions we can write the total Hamiltonian of our model as[1]

$$\begin{aligned} H &= H_0 + H' - \mathscr{E}_0 \\ H_0 &= \sum_{\alpha=1}^{3} \int d^3r \, \frac{1}{2} [\dot{\phi}_\alpha^2 + (\nabla\phi_\alpha)^2 + \mu^2\phi_\alpha^2] \\ &= \sum_{\alpha=-1}^{1} \int d^3k \, a^{\dagger\alpha}(\mathbf{k}) a^\alpha(\mathbf{k})\omega = \sum_{l,m,\alpha} \int_0^\infty dk \, \omega a_l^{\dagger\alpha m}(k) a_l^{\alpha m}(k) \\ H' &= \frac{f}{\mu} \int d^3r \, \rho(r)\tau_\alpha \boldsymbol{\sigma} \cdot \nabla\phi_\alpha(r) \\ &= \frac{f}{\mu} i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \rho(k)\tau_\alpha\boldsymbol{\sigma}\cdot\mathbf{k} \, \frac{a_\alpha(k) - a_\alpha^\dagger(k)}{(2\omega)^{\frac{1}{2}}} \\ &= \frac{f}{\mu} \sum_{m,\alpha} \int_0^\infty \frac{dk \, k^2}{(12\pi^2\omega)^{\frac{1}{2}}} \rho(k)\tau^\alpha\sigma^m [a_1^{\alpha m}(k) + a_1^{\dagger\alpha m}(k)] \end{aligned} \tag{15.5}$$

The constant $\mathscr{E}_0$ has been introduced, as in the Lee model, so that the physical ground-state energy can be adjusted to be zero. This simplifies the formulas which will be given in subsequent chapters and has no physical consequences. It should be noted that H' involves only the field variables with $l = 1$. The other variables occur only in H_0 and hence describe free particles only.

The commutation rules at equal times are the usual ones

$$\begin{gathered} [\phi_\alpha(\mathbf{r},t), \dot{\phi}_\beta(\mathbf{r}',t)] = i\delta_{\beta\alpha} \, \delta^3(\mathbf{r} - \mathbf{r}') \\ [\phi,\phi] = [\dot{\phi},\dot{\phi}] = [\tau,\phi] = [\tau,\dot{\phi}] = [\sigma,\phi] = [\sigma,\dot{\phi}] = [\sigma,\tau] = 0 \quad \text{at equal times} \\ [\sigma_m,\sigma_n] = 2i\sigma_l\epsilon_{lmn} \\ [\tau_\alpha,\tau_\beta] = 2i\epsilon_{\alpha\beta\gamma}\tau_\gamma \end{gathered} \tag{15.6}$$

$\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric tensor which was introduced earlier.

The equations of motion for the meson-nucleon system follow from (15.5) and (15.6) according to the general equation $\dot{\mathcal{O}} = i[H,\mathcal{O}]$:

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \Delta + \mu^2\right) \phi_\alpha(\mathbf{r},t) &= \frac{f}{\mu} \tau_\alpha(t)\boldsymbol{\sigma}(t) \cdot \nabla\rho(\mathbf{r}) \\ \dot{a}_\alpha(\mathbf{k},t) &= -ia_\alpha(\mathbf{k},t)\omega - \frac{f}{\mu} \frac{\rho(k)}{(2\pi)^{\frac{3}{2}}} \frac{\boldsymbol{\sigma}(t)\cdot\mathbf{k}\tau_\alpha(t)}{(2\omega)^{\frac{1}{2}}} \\ \dot{a}_1^{\alpha m}(k,t) &= -ia_1^{\alpha m}(k,t)\omega - i\frac{f}{\mu} \frac{\rho(k)k^2}{(12\pi^2\omega)^{\frac{1}{2}}} \tau^\alpha(t)\sigma^m(t) \end{aligned} \tag{15.7}$$

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[1] As usual, we assume that H_0 is ordered and restrict ourselves to real sources $\rho(r)$. The operators a^α are defined similarly to ϕ^α.

$$\dot{\tau}_\alpha(t) = -\frac{f}{\mu}\epsilon_{\alpha\beta\gamma}\,2\int d^3r\;\phi_\beta(\mathbf{r},t)\tau_\gamma(t)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\rho(r)$$

$$= 2i\;\frac{f}{\mu}\epsilon_{\alpha\beta\gamma}\int\frac{d^3k\;\rho(k)}{(2\pi)^{\frac{3}{2}}}\,\boldsymbol{\sigma}\cdot\mathbf{k}\;\frac{a_\beta(\mathbf{k}) - a_\beta^\dagger(\mathbf{k})}{(2\omega)^{\frac{1}{2}}}\,\tau_\gamma \tag{15.8}$$

$$\dot{\sigma}_j(t) = -\frac{f}{\mu}\epsilon_{jmn}\,2\int d^3r\;\tau_\alpha(t)\phi_\alpha(\mathbf{r},t)\sigma_n\nabla_m\rho(r)$$

$$= 2i\;\frac{f}{\mu}\epsilon_{jmn}\int\frac{d^3k\;\rho(k)}{(2\pi)^{\frac{3}{2}}(2\omega)^{\frac{1}{2}}}\,k_m\sigma_n[a_\alpha(\mathbf{k}) - a_\alpha^\dagger(\mathbf{k})]\tau_\alpha \tag{15.9}$$

Equations (15.7) can be rewritten as an integral equation in our standard way:

$$\phi_\alpha(\mathbf{r},t) = \phi_\alpha^{\text{in}}(\mathbf{r},t) + \frac{f}{\mu}\int d^3r'\;dt'\;\Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}',\,t-t')\tau_\alpha(t')\boldsymbol{\sigma}(t')\cdot\boldsymbol{\nabla}\rho(\mathbf{r}')$$

$$\text{or}\quad a_\alpha(\mathbf{k},t) = A_\alpha(\mathbf{k},t) - \frac{f}{\mu}\int_{-\infty}^{t} dt'\;e^{-i\omega(t-t')}\,\frac{\rho(k)\boldsymbol{\sigma}(t')\cdot\mathbf{k}\tau_\alpha(t')}{(2\pi)^{\frac{3}{2}}(2\omega)^{\frac{1}{2}}} \tag{15.10}$$

The invariance properties introduced into the Hamiltonian permit us to write down several integrals of the equations of motion. That these quantities are constant can be verified by the use of the equations of motion or by checking that they commute with H. However, with the amount of sophistication the reader should have acquired by now, we shall go directly to the heart of the matter and recognize these quantities as the generators of the transformations which leave H invariant. Conservation of angular momentum stems from the invariance of H under simultaneous rotations of $\mathbf{r}$ and $\boldsymbol{\sigma}$. The generator for the transformation $\phi(\mathbf{r},t) \to \phi(\mathbf{r}',t)$, where $\mathbf{r}'$ is related to $\mathbf{r}$ by a rotation about an axis through the origin of coordinates, is the same as for the free fields. A rotation of $\boldsymbol{\sigma}$ through an angle θ about some axis $\mathbf{n}$ is generated by

$$\begin{gathered} U = e^{i\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2} \\ U^{-1}\boldsymbol{\sigma}U = \boldsymbol{\sigma}' \end{gathered} \tag{15.11}$$

For an infinitesimal rotation through an angle $\delta\theta$, U is $1 + i\boldsymbol{\sigma}\cdot\mathbf{n}\,\delta\theta/2$, and the total angular momentum, which is conserved, is [compare (5.3)]

$$\mathbf{J} = \mathbf{S} + \mathbf{L} = \frac{\boldsymbol{\sigma}}{2} - \sum_\alpha\int d^3r\;\dot{\phi}_\alpha(\mathbf{r})\mathbf{r}\times\boldsymbol{\nabla}\phi_\alpha(\mathbf{r})$$

$$= \frac{\boldsymbol{\sigma}}{2} - i\sum_\alpha\int d^3k\;a_\alpha^\dagger(\mathbf{k})\mathbf{k}\times\boldsymbol{\nabla}_{\mathbf{k}}a_\alpha(\mathbf{k}) \tag{15.12}$$

In exactly the same way, the generator of the simultaneous isospin rotations of ϕ_α and τ_α [compare (7.26)] is found to be

$$T_\alpha = \frac{1}{2}\tau_\alpha + t_\alpha = \frac{\tau_\alpha}{2} + \epsilon_{\alpha\beta\gamma}\int d^3r\ \dot{\phi}_\beta(\mathbf{r})\phi_\gamma(\mathbf{r})$$

$$= \frac{\tau_\alpha}{2} + \epsilon_{\alpha\beta\gamma} i \int d^3k\ a_\beta^\dagger(\mathbf{k})a_\gamma(\mathbf{k}) \tag{15.13}$$

Regarding the other classical constants, we have energy conservation but no momentum conservation. This comes about since no time origin is distinguished but the nucleon is fixed in the coordinate origin. Furthermore, we have parity conservation, since our Hamiltonian does not distinguish between a right-handed and left-handed coordinate system. The explicit expression is [see (5.14*c*)]

$$\mathscr{P}_- = \exp\left[-i\pi \sum_{l,m,k} (l+1)a^\dagger_{klm}a_{klm}\right]$$

$$\mathscr{P}_-\phi(\mathbf{r})\mathscr{P}_-^{-1} = -\,\phi(-\mathbf{r}) \qquad \mathscr{P}_-\boldsymbol{\sigma}\mathscr{P}_-^{-1} = \boldsymbol{\sigma} \qquad \mathscr{P}_-\boldsymbol{\tau}\mathscr{P}_-^{-1} = \boldsymbol{\tau} \tag{15.14}$$

Finally, there is one constant connected with a peculiar symmetry of this model which is not present in a more realistic extension of the theory. If we forget about the $l \neq 1$ mesons and look at the reduced Hamiltonian

$$H = \int_0^\infty dk \sum_{m,\alpha}\left\{\omega a_1^{\dagger\alpha m}(k)a_1^{\alpha m}(k) + \frac{f}{\mu}\frac{k^2\rho(k)}{(12\pi^2\omega)^{\frac{1}{2}}}\tau^\alpha\sigma^m[a_1^{\alpha m}(k) + a_1^{\dagger\alpha m}(k)]\right\}$$

we notice that it is invariant under the exchange $\tau^\alpha \leftrightarrow \sigma^m$ and $a_1^{\alpha m}(k) \leftrightarrow a_1^{m\alpha}(k)$. This exchanges J and T and leaves the commutation relations invariant. Consequently, there must be a constant unitary operator effecting this transformation. We leave it to the reader to find an explicit expression for this operator, since it is not very useful. The general consequence of this symmetry is that, for every eigenstate of H belonging to certain eigenvalues J' and T' of J and T, there is a degenerate state with eigenvalues T' and J'. Similarly, the pion-nucleon-scattering phase shift for the state $T = \frac{1}{2}$, $J = \frac{3}{2}$ is equal to that for $T = \frac{3}{2}$, $J = \frac{1}{2}$.

15.3. Comparison with Other Models. If we drop the requirement of a linear coupling, then there are many forms which have the desired invariance properties; a simple one is, perhaps, a quadratic coupling of the form

$$L' = f' \sum_\alpha \int d^3r\ \rho(r)\phi_\alpha(\mathbf{r})\phi_\alpha(\mathbf{r}) \tag{15.15}$$

or

$$L' = f'' \sum_{\alpha\beta\gamma} \int d^3r\ \rho(r)\dot{\phi}_\alpha(\mathbf{r})\phi_\beta(\mathbf{r})\epsilon_{\alpha\beta\gamma}\tau_\gamma \tag{15.16}$$

For a point source, (15.15) reduces to the pair theory, whereas (15.16) is of more complicated mathematical structure. It involves the charge degrees of freedom of the source and does not yield an exact treatment. We shall always assume that such couplings are only small corrections to the leading term (15.2). Terms of the form (15.15) and (15.16) are actually obtained when one attempts to make a nonrelativistic approximation to a relativistic pion-nucleon interaction.[1] Experiments dictate, furthermore, that pions interact with nucleons in S states, as follows from (15.15), but this interaction is considerably weaker than (15.2), although there are no convincing a priori arguments why this should be true.

The fact that in the model with L' given by (15.2) only angular-momentum $l = 1$ or P-wave mesons are coupled results in some important qualitative differences from the previous examples with S waves. Classically, incoming mesons with total linear momentum $\mathbf{k}$ and angular momentum 1 pass the nucleon at a distance k^{-1}. For momenta $k \leq \mu$ this distance is certainly larger than the source radius. Naïvely, one might expect that mesons of this energy cannot be emitted by a nucleon, since they emerge from a part of space where there is no meson source. Quantum-mechanically, the mesons in the source are not sharply localized, but there is a preference for those mesons which, classically, come from near the source. It will turn out that the probability for emission (or absorption) of a meson is proportional to k^2. This is to be anticipated if we look at H', which contains $\rho(k)\mathbf{k}$ and not only $\rho(k)$. Physically, the k^2 dependence comes about as follows. In the direction of emission, mesons of angular momentum 1 appear to emerge from the circumference of a circle centered about the nucleon and having a radius k^{-1}. Quantum-mechanically, these mesons come from an area of order k^{-2}. The interaction strength is proportional to that fraction of the area which is inside the source. If we assume that the source radius $\sim M^{-1}$, where M is the mass of the nucleon, then that fraction is $(k/M)^2$. This energy dependence of the effective interaction strength is reflected, for example, in the momentum distribution of virtual mesons in the nucleon cloud, which is proportional to k^2/ω^3, rather than to $1/\omega^3$, as in the scalar theory. Similarly, in pion-nucleon scattering the low-energy cross section will be proportional to k^4, since it involves an emission and an absorption. This is shown in Fig. 15.1 and is in striking contrast to γ-e scattering, which starts out as a constant, being caused by an S-wave emission and absorption.

The increase of the interaction strength occurs only for $k^{-1} >$ source radius ($\sim M^{-1}$), as shown in Fig. 15.2. Hence the momentum cutoff

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[1] See F. J. Dyson, *Phys. Rev.*, **73**:929 (1948); S. D. Drell and E. M. Henley, *Phys. Rev.*, **88**:1053 (1952); and L. L. Foldy, *Phys. Rev.*, **84**:168 (1951).

$k_{\max} \sim M$ of the Fourier transform of the source determines the maximum strength of the source, which is

$$\max\left[\frac{f^2}{\mu^2}k^2\rho^2(k)\right] \approx \frac{f^2}{\mu^2}k_{\max}^2 \sim f^2\left(\frac{M}{\mu}\right)^2$$

If the size of the source is reduced, then the maximum source strength is increased. Dimensionally, one would expect that an expansion in f is actually one in powers of $fk_{\max}/\mu$. This is indeed the case, and it

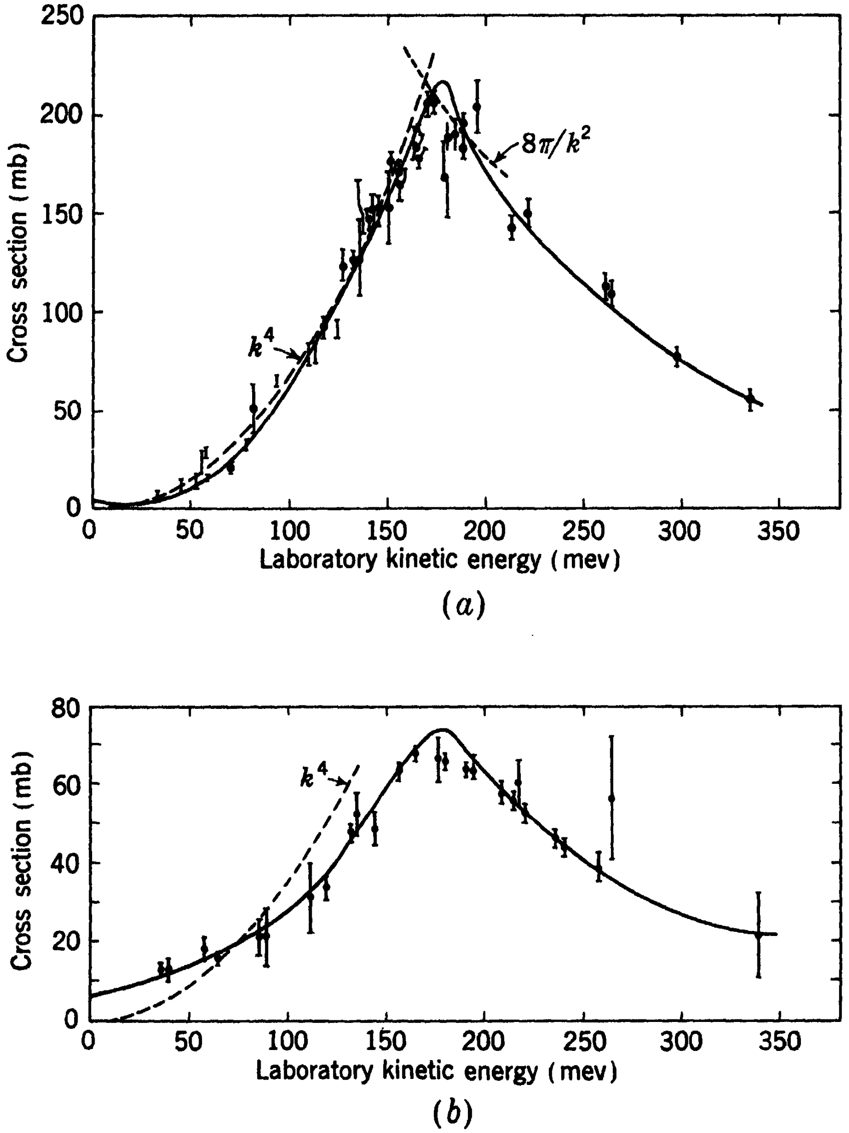

Fig. 15.1. Parts *a* and *b* show the total scattering cross sections for π^+ and π^- mesons on protons, respectively, as a function of the laboratory kinetic energy. The experimental curve and points are taken from H. L. Anderson, W. C. Davidon, and U. E. Kruse, *Phys. Rev.*, **100**:339 (1955). The data stem from the work of many physicists, references to which appear in the above article. The dashed curve superimposed on the low-energy data is proportional to the fourth power of the center-of-mass momentum k. In part *a* there also appears a curve for $8\pi/k^2$, which will be referred to in a subsequent chapter.

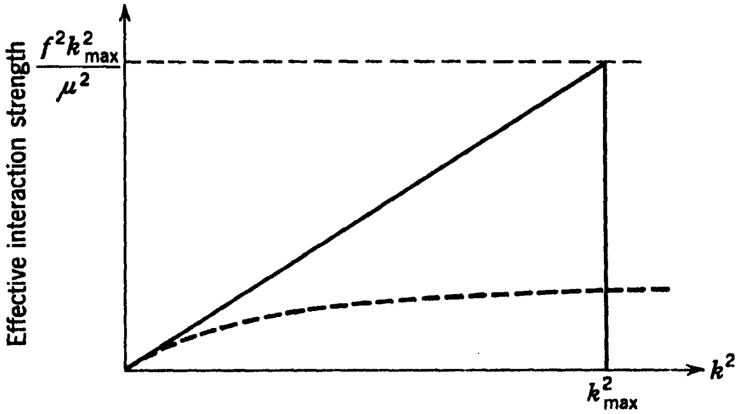

Fig. 15.2. Plot of the effective strength of the pion-nucleon interaction as a function of the square of the pion momentum k. The solid line corresponds to a square cutoff and the dashed line to a Yukawa cutoff.

corresponds to an energy average of the interaction strength. Henceforth, when we talk of a strong or a weak coupling, we shall mean one for which $fk_{max}/\mu \gg 1$ or $\ll 1$, respectively. Regarding the actual form for $\rho(r)$, we shall use a function which goes smoothly to zero outside the range, $1/k_{max}$. Since the detailed shape of $\rho(r)$ is probably without physical significance, we shall be most interested in results which do not depend on it. We shall see that at low energies the observable quantities depend on only one parameter of the source other than its strength, namely, its range. In practice, it is convenient to use a square cutoff for the source in momentum space:

$$\rho(k) = \begin{cases} 1 & \text{for } k < k_{max} \\ 0 & \text{for } k > k_{max} \end{cases}$$

where $k_{max} \sim 1/(\text{source radius}) \sim M$. When the discontinuity of the above choice leads to trouble, one may use a Yukawa cutoff,

$$\rho(k) = \frac{k^2_{max}}{k^2 + k^2_{max}}$$

$$\rho(r) = \frac{e^{-k_{max}r}}{4\pi k_{max}r}$$

or a gaussian cutoff,

$$\rho(k) = e^{-k^2/k^2_{max}}$$

CHAPTER 16

General Features of the Static Model

16.1. Classical Treatment[1] of Stationary Motion. In accordance with our usual procedure we shall first discuss the equations of motion in the limit where all quantities can be treated as commuting numbers once the equations have been obtained.[2] In this case the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are taken to be unit vectors which define the direction of spin and isospin. The equations (15.7) to (15.9) then describe a nonlinear classical system, and a solution can be obtained only in the limit where all quantities oscillate with infinitesimal amplitudes about their equilibrium positions. Rather than solve this problem, we turn to a model with no isospin. In this model the classical equations can be solved exactly, and we obtain more insight into their structure. The reason for neglecting $\boldsymbol{\tau}$ and not $\boldsymbol{\sigma}$ is that omission of the latter would change the dimensionality of the coupling constant and hence the cutoff dependence of the relevant quantities.

In this section we consider solutions with $\phi^{\text{in}} = 0$, e.g., the interaction of the nucleon with its own meson field. Some analogies to the corresponding problem in electromagnetic theory arise. However, the meson field is coupled to $\boldsymbol{\sigma}$ and hence to the rotational rather than the translational degrees of freedom. Thus the main effect of the proper electromagnetic field of a charged body is to generate an addition to the inertial mass, whereas the proper meson field generates a moment of inertia of the nucleon. Analogous to the solution of the electromagnetic equations for which the field follows a uniformly moving charge, solutions will be found where $\boldsymbol{\sigma}$ rotates uniformly and is followed by the meson field. The two theories behave differently in quantum mechanics, where the momentum remains unquantized and the angular momentum must be a half-integer multiple of $\hbar$. There the uniform rotation can occur only for certain frequencies which are determined by the moment of inertia of the meson field. If the moment of inertia is small, then the energy of this motion becomes large. When this energy is higher than the rest energy of the meson, the nucleon will radiate a meson; that is to say, this excited rotational state of the nucleon becomes unstable against decay into a meson and a nucleon in the ground state. Such a motion corresponds to a damped rotation but can be sustained by taking $\phi^{\text{in}} \neq 0$, as we shall see subsequently. Empirically, such unstable excited levels of the nucleon have actually been discovered, and nucleon spectroscopy is an interesting new branch of physics.

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[1] The development in this section closely follows that of W. Pauli, "Meson Theory of Nuclear Forces," 2d ed., Interscience Publishers, Inc., New York, 1948.

[2] In the classical limit, the equations of motion are obtained from Poisson brackets which have the same value as the corresponding commutators.

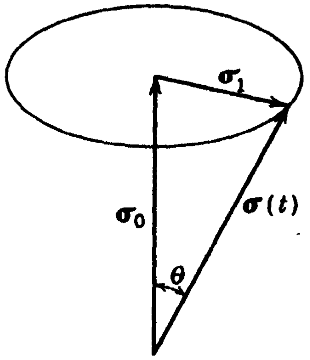

Fig. 16.1. Illustration of the stationary-state solution for the spin $\boldsymbol{\sigma}$.

The variables of our model are $\boldsymbol{\sigma}(t)$, which is a time-dependent unit vector and the classical meson field $\phi(\mathbf{r},t)$. They are connected by the equations

$$\begin{aligned}\phi(\mathbf{r},t) &= \frac{g}{\mu}\int d^3r'\, dt'\, \Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')\boldsymbol{\sigma}(t')\cdot\nabla\rho(\mathbf{r}') \\ &= \frac{g}{\mu}\int \frac{dK_0}{2\pi}\frac{d^3k}{(2\pi)^3}\, dt'\, \frac{e^{-iK_0(t-t')}}{\omega^2-(K_0+i\epsilon)^2}\, e^{i\mathbf{k}\cdot\mathbf{r}}\rho(\mathbf{k})\boldsymbol{\sigma}(t')\cdot i\mathbf{k} \qquad (16.1)\end{aligned}$$

and

$$\dot{\boldsymbol{\sigma}}(t) = -2\frac{g}{\mu}\,\boldsymbol{\sigma}\times\int d^3r'\,\rho(\mathbf{r}')\,\nabla\phi(\mathbf{r}',t) \qquad (16.2)$$

The last equation shows that the meson field produces a torque acting on the spin. The direction of the torque is perpendicular to the gradient of the meson field and to the spin, so that $\boldsymbol{\sigma}^2 = \text{constant}$. If $\nabla\phi$ were a constant, the solution would be a gyration, like the motion of a spinning top in a gravitational field. Although $\nabla\phi$ is not a prescribed constant in our case, we suspect nevertheless that a similar motion is possible. Correspondingly, we seek a solution such that $\boldsymbol{\sigma}$ rotates uniformly around the 3 axis. As an *Ansatz* which contains the angle θ and the frequency as parameters (see Fig. 16.1), we assume

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_1(t)$$

$$\sigma_0 = \begin{pmatrix} 0 \\ 0 \\ \cos\theta \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} \sin\theta\cos\omega_s t \\ \sin\theta\sin\omega_s t \\ 0 \end{pmatrix} \qquad \boldsymbol{\sigma}^2 = 1 \qquad (16.3)$$

Since $\boldsymbol{\sigma}_1$ contains only the frequencies $\pm\omega_s$, the meson field becomes[1]

$$\phi(\mathbf{r},t) = \frac{g}{\mu}\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}}\rho(k)\int_{-\infty}^{\infty}\frac{dt'\,dK_0}{2\pi}\, e^{-iK_0(t-t')}\,\frac{\boldsymbol{\sigma}(t')\cdot i\mathbf{k}}{\omega^2-(K_0+i\epsilon)^2}$$

$$= \frac{g}{\mu}\,[\boldsymbol{\sigma}_0\cdot\nabla V_0(r) + \boldsymbol{\sigma}_1\cdot\nabla V_{\omega_s}(r)] \tag{16.4}$$

V being ($V_0 \equiv V_{\omega_s=0}$)

$$V_{\omega_s}(r) = \int d^3r'\rho(r')\,\frac{\exp\,[-(\mu^2-\omega_s^2)^{\frac{1}{2}}|\mathbf{r}-\mathbf{r}'|]}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \tag{16.5}$$

Depending on whether $\omega_s < \mu$ or $\omega_s > \mu$, V_{ω_s} decays exponentially or reaches to infinity. In the former case there is no radiation, and we get a stationary precession with $\phi^{\text{in}} = 0$, which we shall study first.[2] Like most other calculations in classical field theory, a thorough investigation will involve quite a bit of elementary algebra.

Inserting (16.4) in (16.2) and using $\boldsymbol{\sigma}_0\cdot\boldsymbol{\sigma}_1 = 0$ and[3] $\int d^3r\, f(r)\, x_ix_j = \delta_{ij}\int d^3r f(r)\, r^2/3$, we get

$$\dot{\boldsymbol{\sigma}}_1(t) = 2\frac{g^2}{\mu^2}\Big[\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_1(t)\Big] \times \int d^3r\,\nabla\rho(r)\Big[\boldsymbol{\sigma}_0\cdot\nabla V_0(r) + \boldsymbol{\sigma}_1\cdot\nabla V_{\omega_s}(r)\Big]$$

$$= \boldsymbol{\sigma}_0 \times \boldsymbol{\sigma}_1 C(\omega_s) \tag{16.6}$$

$$C(\omega_s) = \frac{2}{3}\frac{g^2}{\mu^2}\frac{1}{4\pi}\,[f(\omega_s) - f(0)]$$

$$f(\omega_s) = \frac{2}{\pi}\int_0^{\infty}\frac{dk\,k^4\,|\rho(k)|^2}{\omega^2-\omega_s^2}$$

Our *Ansatz* (16.3) implies that $\dot{\boldsymbol{\sigma}}(t) = (\omega_s/\cos\theta)\boldsymbol{\sigma}_0 \times \boldsymbol{\sigma}_1$ and hence solves (16.6), provided

$$\omega_s = C(\omega_s)\cos\theta \tag{16.7}$$

Equation (16.7) determines the frequency ω_s once the angle θ is given. In the following we shall see the circumstances in which (16.7) has solutions for $\omega_s < \mu$. To this end we have to compute the function $C(\omega_s)$.

If the cutoff $k_{\max}$ is much larger than μ, then we can show that the leading term of $f(\omega_s)$ is ω-independent by using the identity

$$\frac{k^4}{k^2+\mu^2-\omega_s^2} = k^2 - (\mu^2-\omega_s^2) + \frac{(\mu^2-\omega_s^2)^2}{k^2+\mu^2-\omega_s^2} \tag{16.8}$$

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[1] We shall henceforth take $\rho(\mathbf{r})$ to be spherically symmetric and real, so that $\rho(\mathbf{k})$ also has these properties. We call the coupling constant g to distinguish between this "neutral pseudoscalar" and the "symmetric pseudoscalar" case.

[2] In this case the $i\epsilon$ is irrelevant and will be dropped.

[3] See footnote 1 on page 24 for an explanation of the notation.

The first term in this decomposition gives the most divergent contribution for $k_{\max} \to \infty$,

$$N = \frac{2}{\pi} \int_0^\infty dk\, k^2\, |\rho(k)|^2 \sim \frac{2}{3} \frac{k_{\max}^3}{\pi} \tag{16.9}$$

whereas the next one diverges only linearly and reflects the radius a of the source,

$$\frac{1}{a} = \frac{2}{\pi} \int_0^\infty dk\, |\rho(k)|^2 = \int d^3r\, d^3r'\, \frac{\rho(r)\rho(r')}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} k_{\max} \tag{16.10}$$

The remaining term converges, and we can therefore set $\rho(k) \equiv 1$. Collecting the contributions, we get

$$f(\omega_s) = N - \frac{\mu^2 - \omega_s^2}{a} + (\mu^2 - \omega_s^2)^{\frac{3}{2}}$$

and

$$C(\omega_s) = \frac{2}{3} \frac{g^2}{\mu^2} \frac{1}{4\pi} \left[\frac{\omega_s^2}{a} + (\mu^2 - \omega_s^2)^{\frac{3}{2}} - \mu^3 \right] \tag{16.11}$$

Our calculation simplifies greatly if the size of the source is much smaller than the Compton wavelength of the pion, e.g., $1/a \gg \mu$.[¶] In this case the field contains mainly high-momentum components, and it can easily follow the comparatively slow motion of the spin. Expanding in successive retardation effects by using

$$\Delta^{\mathrm{ret}}(r,t) = \left[\delta(t) + \delta''(t) \frac{\partial}{\partial \mu^2} + \cdots \right] \frac{e^{-\mu r}}{4\pi r}$$

we get for (16.4) the approximate form

$$\phi(\mathbf{r},t) = \frac{g}{\mu} \left[\boldsymbol{\sigma}(t) + \boldsymbol{\sigma}''(t) \frac{\partial}{\partial \mu^2} + \cdots \right] \cdot \nabla V_0(r) \tag{16.12}$$

and hence the equation

$$\dot{\boldsymbol{\sigma}} = -\frac{g^2}{6\pi\mu^2} \frac{\boldsymbol{\sigma} \times \ddot{\boldsymbol{\sigma}}}{a} \tag{16.13}$$

This is the corresponding limiting form of (16.6) which implies

$$\omega_s = 0 \qquad \text{or} \qquad \omega_s = \frac{6\pi\mu^2 a}{g^2 \cos\theta} \tag{16.14}$$

Thus for μa sufficiently small there always exists a solution with $\omega_s < \mu$. This is no longer true in quantum theory, where the angular momentum has to be half an integer. From what we learn about the motion of a rigid body in quantum theory, it is to be expected that we can obtain the quantum theoretic result if we supplement our classical calculation by the condition that the angular momentum is quantized. We shall see in the following chapter how this is borne out in quantum field theory. To carry out the correspondence, we must first calculate the energy and the angular momentum of our solution by using the formulas for H and $\mathbf{J}$ of the last chapter, but with omission of the isospin. With (16.4) we find, after some calculation,[1]

[¶] The reason why we were not content to consider only this limiting case is that the complete solution is needed for an understanding of the scattering problem.

$$H = \int d^3r \left\{ \tfrac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + \mu\phi^2] + \frac{g}{\mu} \rho\boldsymbol{\sigma} \cdot \nabla\phi \right\} - \mathscr{E}_0$$

$$= \int \frac{d^3k}{(2\pi)^3} \rho^2(k) \frac{g^2}{\mu^2} \frac{k^2}{3} \left\{ \sigma_0^2 \left(\frac{\omega^2}{2\omega^4} - \frac{1}{\omega^2} \right) + \sigma_1^2 \left[\frac{\omega_s^2 + \omega^2}{2(\omega^2 - \omega_s^2)^2} - \frac{1}{\omega^2 - \omega_s^2} \right] \right\} - \mathscr{E}_0$$

$$= -\frac{g^2}{4\pi\mu^2} \frac{1}{6} \left\{ f(0)\cos^2\theta + \left[f(\omega_s) - 2\omega_s^2 \frac{\partial}{\partial(\omega_s)^2} f(\omega_s) \right] \sin^2\theta \right\} - \mathscr{E}_0 = \Delta E$$

$$\mathscr{E}_0 = -\frac{g^2}{4\pi\mu^2} \frac{f(0)}{6} \tag{16.15}$$

$$\Delta E = \frac{g^2}{4\pi\mu^2} \frac{\sin^2\theta}{6} \left[\frac{\omega_s^2}{a} + \mu^3 - (\mu^2 - \omega_s^2)^{\frac{3}{2}} - 3\omega_s^2(\mu^2 - \omega_s^2)^{\frac{1}{2}} \right]$$

The energy $\mathscr{E}_0$ was adjusted to be the energy[2] of the static solution ($\omega_s = 0$). The remainder we called ΔE, since it is the energy difference between the rotating and the static solution. We note that, for sufficiently large a, $\Delta E > 0$; that is, the static solution is the one with lowest energy.

For the angular momentum (15.12), we find

$$\mathbf{J} = \frac{\boldsymbol{\sigma}}{2} + \frac{g^2}{\mu^2} \int \frac{d^3k}{(2\pi)^3} \frac{\rho^2(k)\dot{\boldsymbol{\sigma}}_1 \cdot \mathbf{k}}{(\omega^2 - \omega_s^2)} \mathbf{k} \times \nabla_k \left(\frac{\boldsymbol{\sigma}_0 \cdot \mathbf{k}}{\omega^2} + \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{k}}{\omega^2 - \omega_s^2} \right)$$

$$= \frac{\boldsymbol{\sigma}}{2} + \frac{g^2}{\mu^2} \int \frac{d^3k}{(2\pi)^3} \frac{k^2\rho^2(k)}{3(\omega^2 - \omega_s^2)} \left(\frac{\dot{\boldsymbol{\sigma}}_1 \times \boldsymbol{\sigma}_0}{\omega^2} + \frac{\dot{\boldsymbol{\sigma}}_1 \times \boldsymbol{\sigma}_1}{\omega^2 - \omega_s^2} \right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \left\{ \cos\theta + \omega_s \frac{g^2}{4\pi\mu^2} \frac{2}{3} \sin^2\theta \left[\frac{1}{a} - \frac{3}{2}(\mu^2 - \omega_s^2)^{\frac{1}{2}} \right] \right\} \tag{16.16}$$

The components of $\mathbf{J}$ parallel to $\boldsymbol{\sigma}_1$ cancel because of (16.6), as they should. Hence the total angular momentum is a constant vector in the z direction.

---

[1] Note that the positive contributions come from H_0 and the (larger) negative contributions from H'.

[2] This turns out to be independent of θ, as it should. The energy of the ground state is zero.

To get simpler results, we go to the limit $a^{-1} \gg \mu$, which is actually satisfied in the pion-nucleon interaction. We then have

$$\omega_s = \frac{3a}{2 \cos \theta (g^2/4\pi\mu^2)}$$

and hence

$$\mathbf{J} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \sec \theta \end{bmatrix} \qquad |J| > \tfrac{1}{2} \tag{16.17}$$

For ΔE expressed in terms of $\mathbf{J}$, we find

$$\Delta E = \frac{4\pi}{g^2} \frac{3a\mu^2}{2} (\mathbf{J}^2 - \tfrac{1}{4}) \qquad \omega_s = \frac{\partial \Delta E}{\partial J} \tag{16.18}$$

corresponding to the rotational energy of a rigid body with a moment of inertia $g^2/(4\pi\mu^2 3a)$. The latter is due to the meson field following the spin. Its magnitude can be easily understood and is the sum over all momenta of

(Probability for finding a meson) × (energy of the meson) × (distance it can escape from the source)^2 =

$$\int \frac{d^3k}{(2\pi)^3} \left[\frac{g^2}{\mu^2} \frac{\rho^2(k)k^2}{\omega^3}\right] \omega \left(\frac{1}{\omega^2}\right)$$

The expression (16.18) can be used to estimate the energy of the next level in quantum theory.[1] The one after the ground state ($J = \frac{1}{2}$) is a $J = \frac{3}{2}$ level with an energy

$$\Delta E_{\frac{3}{2}} = \frac{6\pi a\mu^2}{g^2} = \omega_b$$

Thus, only for sufficiently large $g^2/4\pi$ can we meet the condition $\omega_b < \mu$ for this state not to decay by meson radiation. The reason for this is that only for sufficiently strong coupling is the meson cloud thick enough to have an appreciable moment of inertia. For too small a moment of inertia the rotation has to be very fast to acquire one unit of angular momentum. The meson cloud will then not endure the centrifugal force and will break up. This situation is analogous to the neutron-proton interaction, which is not strong enough to bind more than the triplet-spin S state (deuteron). Higher-angular-momentum states are torn apart by the centrifugal force. In nature we seem to have this situation; that is, the coupling is not strong enough for a stable excited state of the nucleon. However, experimentally one finds a huge resonance in the scattering of mesons for meson energies $\sim 2\mu$.

---

[1] There $J^2 - \frac{1}{4}$ should be replaced by $J(J + 1) - \frac{3}{4}$.

This is attributed to an unstable excited state, whose influence on scattering will be studied in the next section.

16.2. Classical Treatment of Scattering. For $\omega_b > \mu$, $C(\omega_b)$ as given by (16.11) becomes complex. Correspondingly, our *Ansatz* (16.3) no longer works. A complex frequency implies a damping due to radiation. To sustain the motion, we introduce an incoming field ϕ^{in}, realized by an incident beam of mesons, and the precession then shows up as a resonance in the scattering.

For ϕ^{in} we take a plane wave

$$\phi^{\text{in}}(\mathbf{r},t) = Ae^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)} + A^*e^{-i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)} \tag{16.19}$$

the amplitude of which is of the order (volume of normalization)$^{-\frac{1}{2}}$, so that we can neglect powers higher than the first power of A. It is now convenient to decompose $\boldsymbol{\sigma}$ as follows:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_1(t) \qquad \boldsymbol{\sigma}_1(t) = \boldsymbol{\sigma}(\omega_0)e^{-i\omega_0 t} + \boldsymbol{\sigma}^*(\omega_0)e^{+i\omega_0 t} \tag{16.20}$$

Since we want a solution for $\boldsymbol{\sigma}$ which is static before the incident wave arrives,[1] $\boldsymbol{\sigma}_1$ will be of the order A, and we may drop higher powers of it. The representation of the $\boldsymbol{\sigma}$'s corresponding to the *Ansatz* (16.3) is

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \boldsymbol{\sigma}(\omega_0) = \frac{\theta}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \tag{16.21}$$

The treatment of the field equation with ϕ^{in},

$$\phi(\mathbf{r},t) = \phi^{\text{in}}(\mathbf{r},t) + \frac{g}{\mu}\int d^3r'\,dt'\,\Delta^{\text{ret}}(\mathbf{r}-\mathbf{r}', t-t')\boldsymbol{\sigma}(t')\cdot\boldsymbol{\nabla}\rho(\mathbf{r}') \tag{16.22}$$

is completely analogous to the previous calculation, and we immediately turn to what corresponds to the positive-frequency part of (16.6):

$$i\omega_0\boldsymbol{\sigma}(\omega_0) = 2i\frac{g}{\mu}A\boldsymbol{\sigma}_0 \times \mathbf{k}_0\rho(k) + C(\omega_0)\boldsymbol{\sigma}_0 \times \boldsymbol{\sigma}(\omega_0) \tag{16.23}$$

The first term on the right-hand side is the torque due to the incident field. The second term is the reaction of the field of the source, and it is exactly the same expression we had before. From (16.23) it appears that only the component of $\mathbf{k}_0$ perpendicular to $\boldsymbol{\sigma}_0$ contributes to the motion. Putting the x axis in this direction, we find that the first term is in the y direction and the second is in the xy plane perpendicular to $\boldsymbol{\sigma}(\omega_0)$. Therefore we can satisfy (16.23) with the *Ansatz*

$$\boldsymbol{\sigma}(\omega_0) = \gamma\boldsymbol{\sigma}_0 \times \mathbf{k}_0 - \beta\boldsymbol{\sigma}_0 \times (\boldsymbol{\sigma}_0 \times \mathbf{k}_0) \tag{16.24}$$

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[1] To realize this situation, we have to form a wave packet for ϕ^{in}.

or, in the above-mentioned frame,

$$\boldsymbol{\sigma}(\omega_0) = |\mathbf{k}_0| \begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix}$$

Substituting into (16.23), we find

$$\gamma = \frac{2gA\omega_0\rho(k)}{\mu[\omega_0^2 - C^2(\omega_0)]} \qquad \beta = \frac{-2igAC(\omega_0)\rho(k)}{\mu[\omega_0^2 - C^2(\omega_0)]} \tag{16.25}$$

where $C(\omega_0)$ is

$$C(\omega_0) = \frac{g^2}{6\pi\mu^2}\left(\frac{\omega_0^2}{a} - ik_0^3 - \mu^3\right) \tag{16.26}$$

The motion of $\boldsymbol{\sigma}$ is similar to that of a harmonic oscillator with damping and under the influence of a periodic external force. Neglecting C, we have a linear motion in the y direction which is in phase with the torque from ϕ^{in}. C represents the influence of the proper field which generates a torque in the x direction for this motion. The result is an elliptic motion whose projection in the y direction is out of phase with ϕ^{in}, as is indicated by the imaginary part of C (Fig. 16.2). If C becomes so large that $\operatorname{Re}[\omega_0^2 - C^2(\omega_0)] = 0$, the motion in the y direction is 90° out of phase. If $1/a \gg \omega_0$, this happens for

$$\omega_r = 6\pi\mu^2 a/g^2$$

and the amplitude of the oscillation attains its maximum value at this energy. This resonance frequency is just the one for which we found the precessing solution with $\cos\theta = 1$.

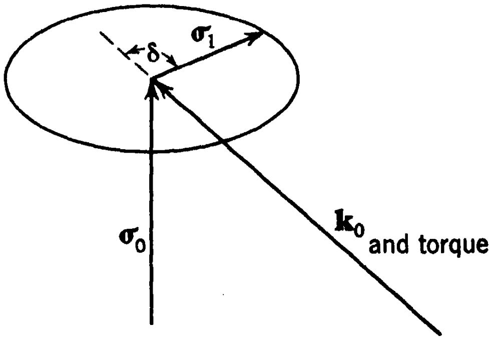

Fig. 16.2. Illustration of the solution for the spin $\boldsymbol{\sigma}$ under the influence of an incident meson wave of momentum $\mathbf{k}_0$. The phase between the precessing spin $\boldsymbol{\sigma}_1$ and $\mathbf{k}_0$ is characterized by δ.

To find the scattering cross section, we evaluate $\phi(\mathbf{r},t)$, given by (16.22) in the limit $r \to \infty$. $\boldsymbol{\sigma}_0$ creates only an exponentially decaying field, and at large distances the oscillating field produced by $\boldsymbol{\sigma}(\omega_0)$ is the usual outgoing radial wave (c.c. = complex conjugate),

$$\begin{aligned} \phi &= Ae^{i(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t)} + \frac{ig\rho(k_0)e^{i(k_0 r - \omega_0 t)}}{4\pi\mu r}\boldsymbol{\sigma}(\omega_0)\cdot\mathbf{k}_s + \text{c.c.} \\ &= Ae^{-i\omega_0 t}\left[e^{i\mathbf{k}_0\cdot\mathbf{r}} + \frac{e^{ik_0 r}}{r}f(\vartheta)\right] + \text{c.c.} \end{aligned} \tag{16.27}$$

where $\mathbf{k}_s$ has the length of $\mathbf{k}_0$ and the direction of $\mathbf{r}$. The differential cross section is then found in the standard manner by comparing the incident and the scattered flux:

$$\frac{d\sigma}{d\Omega} = |f(\vartheta)|^2 = \left| \frac{g^2 \rho^2(k)}{4\pi\mu^2} \frac{2\omega_0}{\omega_0^2 - C^2(\omega_0)} \right|^2$$

$$\times \left| \mathbf{k}_s \cdot \boldsymbol{\sigma}_0 \times \mathbf{k}_0 + i \frac{C(\omega_0)}{\omega_0} \mathbf{k}_s \cdot \boldsymbol{\sigma}_0 \times (\boldsymbol{\sigma}_0 \times \mathbf{k}_0) \right|^2 \tag{16.28}$$

In contradistinction to the examples we considered in the second part of the book, this cross section is not isotropic. It depends on the relative orientation of the three vectors $\mathbf{k}_0$, $\mathbf{k}_s$, $\boldsymbol{\sigma}_0$ and contains a great deal of information. Mesons are emitted preferentially in the direction of motion of the spin. The motion due to the incident meson gives an angular distribution peaked perpendicular to $\boldsymbol{\sigma}_0$ and $\mathbf{k}_0$, whereas the other term of the amplitude which originates from the field reaction favors the incident direction. All this is valuable for checking the theory experimentally, but these subtle details are not rendered correctly by our classical theory. The reason is that a spin $\frac{1}{2}$ is remote from a classical angular momentum, its zero-point oscillations being of its own magnitude (classically, $\boldsymbol{\sigma}^2 = 1$; quantum-mechanically, $\boldsymbol{\sigma}^2 = 3$). Correspondingly, we shall ignore these predictions of the theory and average over the spin directions:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{4\pi} \int d\Omega_s \, |f|^2$$

$$= \frac{4}{3} \left(\frac{g^2}{4\pi}\right)^2 \frac{k^4}{\mu^4 \omega_0^2} \left| \frac{\omega_0^2}{\omega_0^2 - C^2(\omega_0)} \right|^2 \left[\sin^2 \vartheta + \left| \frac{C(\omega_0)}{\omega_0} \right|^2 \frac{7 \cos^2 \vartheta + 1}{5} \right] \tag{16.29}$$

where k is the length of $\mathbf{k}_0$ and $\mathbf{k}_s$ and ϑ is the angle between them.

If ω_0 and the resonance energy are much less than $\omega_{\max}$, we have

$$C(\omega_0) = \frac{\omega_0^2}{\omega_r} - i \frac{\Gamma}{2} \tag{16.30}$$

with the resonance energy

$$\omega_r = \frac{6\pi a \mu^2}{g^2} \tag{16.31}$$

and the width

$$\Gamma = \frac{4}{3} \frac{g^2}{4\pi\mu^2} k_r^3 \tag{16.32}$$

Near the resonance the cross section assumes the familiar shape

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{8}{15} \left(\frac{g^2}{4\pi}\right)^2 \frac{k^4}{\mu^4} \frac{3 + \cos^2 \vartheta}{(\omega_0 - \omega_r)^2 + \Gamma^2/4} \tag{16.33}$$

This angular distribution is only qualitatively correct (the exact quantum-mechanical result is $1 + 3\cos^2\theta$). At ω_r the total cross section reaches a peak value $\bar{\sigma} = 16\pi/k_r^2$, which is twice the quantum-mechanical value for a $J = \frac{3}{2}$ resonance (see Fig. 15.1). Similarly, this kind of calculation[1] would not render accurately the charge distribution if isospin had been included. The experiments agree within their accuracy of 5 to 10 per cent with the quantum-mechanical predictions for a $J = \frac{3}{2}$ resonance. Thus the classical calculation is not suited for comparison with this kind of measurement. Nevertheless, the dominant fact of low-energy pion physics, namely, a resonant state with higher angular momentum, is correctly predicted by the classical model. Thus the basic features of the pion-nucleon interaction can be understood in terms of the intuitive picture of a nucleon surrounded by its meson field.[2]

16.3. Quantum Aspects of the Static Model. Having discussed the classical limiting form of the equations of motion, we turn our attention to effects originating from the quantum nature of the field and spin and isospin. Whereas spin and isospin are definitely in the region of small quantum numbers, neither of the two complementary aspects of the field is clearly realized in the pion-nucleon system, but both are needed for its understanding. Let us first study the features of the elementary processes described by H'. To obtain some insight into the isospin properties, we use circular components [see (15.4)]. The application of $\phi^\alpha\tau^\alpha$ to a bare nucleon state creates[3] $\pi^0(\phi^0)$, $\pi^+(\phi^1)$, and $\pi^-(\phi^{-1})$ mesons and in the last two cases simultaneously changes a proton into a neutron (τ^-), or vice versa (τ^+), in such a manner as to conserve charge. We obtain[4]

$$\begin{aligned} -(\tfrac{1}{3})^{\frac{1}{2}} \sum_\alpha \phi_\alpha\tau_\alpha \,|\, p) &= (\tfrac{2}{3})^{\frac{1}{2}} \,|\, n\pi^+) - (\tfrac{1}{3})^{\frac{1}{2}} \,|\, p\pi^0) \\ &= |\, T = \tfrac{1}{2}, T_z = \tfrac{1}{2}) \\ -(\tfrac{1}{3})^{\frac{1}{2}} \sum_\alpha \phi_\alpha\tau_\alpha \,|\, n) &= -(\tfrac{2}{3})^{\frac{1}{2}} \,|\, p\pi^-) + (\tfrac{1}{3})^{\frac{1}{2}} \,|\, n\pi^0) \\ &= |\, T = \tfrac{1}{2}, T_z = -\tfrac{1}{2}) \end{aligned} \qquad (16.34)$$

with

$$\begin{aligned} |\, n\pi^+) &\equiv -\phi^1 \,|\, n) \\ |\, p\pi^+) &= -\phi^1 \,|\, p) \end{aligned}$$

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[1] See W. W. Wada, *Phys. Rev.*, **88**:1032 (1952).

[2] In our classical calculation the nucleon has $J = \frac{1}{2}$.

[3] We remind the reader that $\phi^0 = \phi_3$, $\phi^1 = \phi_+$, $\phi^{-1} = \phi_-$.

[4] The factor $(\frac{1}{3})^{\frac{1}{2}}$ is a normalization constant. The relative phases have been chosen to agree with E. U. Condon and G. H. Shortley, "The Theory of Atomic Spectra," chap. 2, Cambridge University Press, New York, 1953.

Since $\sum_\alpha \phi_\alpha \tau_\alpha$ is invariant under rotations in isospin space (e.g., it is a scalar in this space), the states (16.34) must have isospin $\frac{1}{2}$, like the nucleons. In fact, the coefficients of the expressions on the right-hand side of (16.34) are the well-known Clebsch-Gordan coefficients, formed by the addition of the pion isospin ($= 1$) to that of the bare nucleon ($= \frac{1}{2}$), so as to add to a total isospin of $\frac{1}{2}$. The other states of nucleonic charge of the pion-nucleon system must have $T = \frac{3}{2}$, $T_z = \pm\frac{1}{2}$ and must be orthogonal to the $T = \frac{1}{2}$ states; they are[1]

$$\begin{aligned} | T = \tfrac{3}{2}, T_z = \tfrac{1}{2}) &= (\tfrac{1}{3})^{\frac{1}{2}} \, | n\pi^+) + (\tfrac{2}{3})^{\frac{1}{2}} \, | p\pi^0) \\ | T = \tfrac{3}{2}, T_z = -\tfrac{1}{2}) &= (\tfrac{1}{3})^{\frac{1}{2}} \, | p\pi^-) + (\tfrac{2}{3})^{\frac{1}{2}} \, | n\pi^0) \end{aligned} \tag{16.35}$$

The remaining two states of $T = \frac{3}{2}$ are

$$\begin{aligned} | T = \tfrac{3}{2}, T_z = \tfrac{3}{2}) &= | p\pi^+) \\ | T = \tfrac{3}{2}, T_z = -\tfrac{3}{2}) &= | n\pi^-) \end{aligned} \tag{16.36}$$

All the states of (16.34) to (16.36) are eigenstates of T^2 and T_z. We note that the $T = \frac{1}{2}$, $T_z = \pm\frac{1}{2}$ states contain twice as many charged pions as neutral ones, whereas this ratio is reversed in the $T = \frac{3}{2}$, $T_z = \pm\frac{1}{2}$ states. If we average over all $T = \frac{1}{2}$ or $T = \frac{3}{2}$ states, it is easy to see that there are always equal amounts of π^0, π^+, and π^- mesons. This is an expression of the isotropy in isospin space which was built into the form of the Hamiltonian (15.5) by choosing the same-strength coupling constant $|f|$ for all components ϕ_α. From (16.34) it follows that the consequent interaction strength for emission of a charged meson is $(2)^{\frac{1}{2}}$ times that for a neutral one. Furthermore, the coupling constant for the emission of neutral pions has the opposite sign for protons and neutrons. This last statement is of nontrivial group theoretic origin and can be observed experimentally by studying, for example, the elastic (e.g., the final nuclear state has the same energy as the initial one) photoproduction of π^0 from deuterium. The π^0 waves emerging from the proton and neutron will interfere destructively or constructively, depending on the relative sign of the coupling constants. As was mentioned in the last chapter, the momentum-space dependence of H', e.g., $H' \sim k$, is characteristic of a P-state interaction and will play an essential role in the development of the following chapters. Similar to the isospin, $\int d^3r \, \rho(r) \boldsymbol{\sigma} \cdot \nabla\phi$, when applied to a nucleon, only generates states of the same total angular momentum $\frac{1}{2}$. There are again six possible states of the pion-nucleon system, corresponding to $J = \frac{3}{2}$ and $J = \frac{1}{2}$. These can be written in complete analogy to (16.34) and (16.35) if we designate the pion states by their l_z components 1, 0, -1 and the nucleon spin components by $\uparrow$(up) and $\downarrow$(down):

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[1] The phases have again been chosen so as to agree with the standard Clebsch-Gordan coefficients (see, e.g., *ibid.*).

$$\begin{aligned}
|J = \tfrac{3}{2}, J_z = \tfrac{3}{2}) &= |\uparrow 1) \\
|J = \tfrac{3}{2}, J_z = \tfrac{1}{2}) &= (\tfrac{1}{3})^{\frac{1}{2}}\,|\downarrow 1) + (\tfrac{2}{3})^{\frac{1}{2}}\,|\uparrow 0) \\
|J = \tfrac{3}{2}, J_z = -\tfrac{1}{2}) &= (\tfrac{1}{3})^{\frac{1}{2}}\,|\uparrow -1) + (\tfrac{2}{3})^{\frac{1}{2}}\,|\downarrow 0) \\
|J = \tfrac{3}{2}, J_z = -\tfrac{3}{2}) &= |\downarrow -1) \\
|J = \tfrac{1}{2}, J_z = \tfrac{1}{2}) &= (\tfrac{2}{3})^{\frac{1}{2}}\,|\downarrow 1) - (\tfrac{1}{3})^{\frac{1}{2}}\,|\uparrow 0) \\
|J = \tfrac{1}{2}, J_z = -\tfrac{1}{2}) &= -(\tfrac{2}{3})^{\frac{1}{2}}\,|\uparrow -1) + (\tfrac{1}{3})^{\frac{1}{2}}\,|\downarrow 0)
\end{aligned} \tag{16.37}$$

Although we are in the region of small quantum numbers, the classical properties of the angular-momentum additions are already apparent. For instance, in the state $|J = \frac{3}{2}, J_z = \frac{1}{2})$, which classically leans to the side, the $|\uparrow 0)$ configuration has twice the weight of the $|\downarrow 1)$ configuration. In the $|J = \frac{1}{2}, J_z = \frac{1}{2})$ state, on the other hand, the situation is reversed. Because of conservation of angular momentum, only the $J = \frac{1}{2}$ states are accessible, when a nucleon emits a pion, and (16.37) then shows that in this process the nucleon flips its spin two out of three times. This is the quantum-mechanical expression of the classical spin precession we studied earlier.

A statement analogous to the ratio of charged to neutral pions is that the meson cloud has a spherical shape. There are twice as many mesons $l_z = \pm 1$ as with $l_z = 0$ in the $J = \frac{1}{2}$ state. Since the former have a distribution proportional to $\frac{1}{2}\sin^2\theta$ and the latter to $\cos^2\theta$, we obtain an isotropic distribution, conforming with the general theorem that a spin-$\frac{1}{2}$ particle has no quadrupole moment.[1] On the other hand, a $J = \frac{3}{2}, J_z = \pm\frac{1}{2}$ state has a meson distribution $\propto 1 + 3\cos^2\theta$, as was mentioned in the last section.

We conclude this section with some remarks about the eigenvalue spectrum of H. If $H' = 0$, then the spectrum is simple (see, e.g., Fig. 12.3). For $T = J = \frac{1}{2}$, there is the nucleon, a fourfold degenerate state with $E = 0$. At $E = \mu$, the continuum of states of one meson begins, and we can have states with $T = \frac{3}{2}$, $J = \frac{1}{2}$; $T = \frac{1}{2}$, $J = \frac{3}{2}$; $T = \frac{3}{2}$, $J = \frac{3}{2}$. In general, states of mesons coupled to the nucleon with T or $J = (2n + 1)/2$ appear at or above an energy $E = n\mu$.

When H' is switched on, all these energy levels are shifted to lower energies,[1] but it is believed that for sufficiently small H' the structure of the energy spectrum is not changed. This means that there is still a fourfold degenerate ground state $|\,N\rangle$, which we shall call the physical nucleon. After an energy gap of magnitude μ, there will be the continuum of physical "nucleon + one meson" states. These can be obtained by applying $\phi^{\text{in}}(\mathbf{k} \to 0, t)$ to $|\,N\rangle$. Since the operator ϕ^{in} has a time dependence $e^{i\mu t}$, the resulting state is actually an eigenstate of H with energy μ. States with several mesons can be generated by applying products of creation operators to the ground state.

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[1] Although the statement is deduced here only for the case of one meson in the could, it can be generalized to any system of $J = \frac{1}{2}$.

It is to be expected that for sufficiently strong interactions the structure of the energy spectrum changes. For f larger than a certain critical value, we may find discrete states of the meson-nucleon system, representing stable excited states of the physical nucleon. As we saw, in the classical picture these states would correspond to a precession of the nucleon spin (or isospin) and its meson field with frequencies $<\mu$. In the classical approximation this occurs for sufficiently strong coupling constants ($f > \mu a$). In quantum theory no one has found (except in certain approximations) the minimum strength of the coupling constant that is necessary to obtain bound excited states. It has not even been possible to prove that these discrete states will always have $E > 0$, so that the ground state of the system corresponds to $T = J = \frac{1}{2}$. At this point we have to proceed semiempirically and use the fact that the nucleon we find in nature has $J = T = \frac{1}{2}$ and that there are no stable isobars, the first excited state with $J = T = \frac{3}{2}$ level already being in the continuum. Experience with approximate treatments tells us that this kind of behavior is also expected from the model if the empirical coupling constant and cutoff are used. Consequently we shall base our further development on the assumption that the energy spectrum is normal.

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[1] This is the prediction of perturbation theory, since

$$\Delta E = \sum_n \frac{|\langle n\,|\,H'\,|\,0\rangle|^2}{E_0 - E_n}$$

where E_0 and E_n are the energies of the ground and nth states, respectively. We shall see that for the ground state this is an exact statement.

CHAPTER 17

The Ground State

17.1. Exact Results. For a long time it has been one of the main goals of meson theory to analyze the physical nucleon in terms of the bare nucleon and its surrounding meson cloud. This problem led into a dead-end road and has not yet yielded to calculations. Furthermore, even if it could be solved, the result would be of limited value. The reason is that the large effect in pion physics, namely, the resonant state of the nucleon, is not important for the ground state. For it, medium-sized effects included in the model and those excluded are as important as the resonance. As we shall see, the measurable quantities of the ground state are predicted by the model to within an accuracy of only 50 per cent. Thus, with regard to fruitfulness and complexity, the problem can be compared with a calculation of the ground state of light nuclei with inaccurate nuclear forces.

One of the main advances in static meson theory has actually been a divorce from the concern with the physical nucleon. From a practical point of view, the model is most powerful in describing processes such as pion-nucleon scattering. For this, one needs only certain matrix elements between the ground state, and many quantities, such as the self-energy $\mathscr{E}_0$, remain unobservable. Nevertheless, to gain some insight, we shall review in this chapter the important features of what is known about the ground state and what has been learned in the past by various approximation schemes. These methods form the bulk of the later sections.

First we shall study the form of important matrix elements. We have already argued that the ground state

$$|N\rangle = |N, \text{in}\rangle = |N, \text{out}\rangle$$

is fourfold degenerate. The degenerate states all have spin and isospin

$\frac{1}{2}$ and can be written as $|p\uparrow\rangle, |p\downarrow\rangle, |n\uparrow\rangle, |n\downarrow\rangle$. Matrix elements between them can be largely reduced by taking advantage of the transformation properties under various invariance groups. The fact that these states transform like bare nucleons under spatial and isospin rotations can be expressed by[1]

$$\begin{aligned}\langle\alpha, j \,|\, e^{i\mathbf{J}\cdot\mathbf{n}} \,|\, \alpha', j'\rangle &= (\alpha, j \,|\, e^{i\frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{n}} \,|\, \alpha', j') \\ \langle\alpha, j \,|\, e^{i\mathbf{T}\cdot\mathbf{n}'} \,|\, \alpha', j'\rangle &= (\alpha, j \,|\, e^{i\frac{1}{2}\boldsymbol{\tau}\cdot\mathbf{n}'} \,|\, \alpha', j')\end{aligned} \tag{17.1}$$

for arbitrary vectors $\mathbf{n}$ and $\mathbf{n}'$. Here, we have used subscripts $\alpha = 1, 2$ for the isospin and $j = 1, 2$ for the angular-momentum classification of the four nucleon states. Furthermore, since $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ have the same transformation properties under rotations as $\mathbf{J}$ and $\mathbf{T}$ and since our nucleon states are eigenstates of the latter operators, we have

$$\begin{aligned}\langle p \,|\, \tau_3 \,|\, p\rangle &= -\langle n \,|\, \tau_3 \,|\, n\rangle \\ \langle p \,|\, \tau_3 \,|\, n\rangle &= 0\end{aligned} \tag{17.2}$$

By rotations these relations can be generalized to

$$\langle\alpha, j \,|\, \tau_\beta \,|\, \alpha', j'\rangle = \mathfrak{r}_1(\alpha, j \,|\, \tau_\beta \,|\, \alpha', j')\, \delta_{jj'} \tag{17.3}$$

where $\mathfrak{r}_1$ is a number independent of β, α, and α'. Similarly, we have

$$\langle\alpha, j \,|\, \sigma_i \,|\, \alpha', j'\rangle = \mathfrak{r}_1'(\alpha, j \,|\, \sigma_i \,|\, \alpha', j')\, \delta_{\alpha\alpha'} \tag{17.4}$$

where $\mathfrak{r}_1'$ must be the same number as $\mathfrak{r}_1$ because of the invariance of the theory under the exchange of $\mathbf{J}$ and $\mathbf{T}$. Simultaneous rotations in spin and isospin space give

$$\langle\xi' \,|\, \sigma_i\tau_\alpha \,|\, \xi\rangle = \mathfrak{r}_2(\xi' \,|\, \sigma_i\tau_\alpha \,|\, \xi) \tag{17.5}$$

which contains another constant $\mathfrak{r}_2$. To avoid crowding of subscripts, we use a single subscript $\xi = 1, \ldots, 4$ for labeling the nucleon states whenever it is convenient.

So far we have seen that the matrix elements of $\mathbf{J}$, $\mathbf{T}$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ and their products between physical nucleon states involve but two free parameters, $\mathfrak{r}_1$ and $\mathfrak{r}_2$. They are, in fact, not completely free, since they must satisfy several inequalities. To this end, we express the physical nucleon in terms of bare states

$$|\,\xi\rangle = R(a^\dagger, \boldsymbol{\sigma}, \boldsymbol{\tau})\,|\,\xi) \tag{17.6}$$

where the "dressing operator" R is a rotationally invariant combination of the $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, and meson-creation operators, as in (16.34), for example. Since both the physical and bare nucleons have spins and isospins of $\frac{1}{2}$, the angular momentum and the isospin of the meson cloud can only be 0 or 1. If we label the various terms in R by subscripts l, l_z, t, and t_z, in that order, to indicate the quantum numbers of the corresponding part of the meson cloud, then R will be a linear combination of the operators R'_{0000}, R'_{1l_z00}, R'_{001t_z}, and $R'_{1l_z1t_z}$. Thus, we can write the ground state $|p\uparrow\rangle$, for example, as

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[1] The operators $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ are time-dependent but are here taken at $t = 0$, as is always understood when no time dependence is mentioned. Hence, the matrix elements of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ between bare states $|)$ are the standard Pauli matrices.

$$\begin{aligned} |p\uparrow\rangle = R'_{0000}\,|p\uparrow) &+ R'_{0011}\,|n\uparrow) + R'_{0010}\,|p\uparrow) \\ &+ R'_{1100}\,|p\downarrow) + R'_{1000}\,|p\uparrow) + R'_{1111}\,|n\downarrow) \\ &+ R'_{1011}\,|n\uparrow) + R'_{1110}\,|p\downarrow) + R'_{1010}\,|p\uparrow) \end{aligned}$$

where the R' contain only meson-creation operators. So far we have assured only that $|p\uparrow\rangle$ is an eigenstate of T_z and J_z, but in Chap. 16 we learned how to prepare eigenstates of T and J. Following the argument given there, we introduce the operators R_{0000}, R_{1l_z00}, R_{001t_z}, $R_{1l_z1t_z}$ in such a way that those with different l_z (or t_z) are connected by rotations. The R differ from the R' by the extraction of some Clebsch-Gordan coefficients, and we get, in terms of the former,

$$\begin{aligned} |p\uparrow\rangle = R_{0000}\,|p\uparrow) &+ (\tfrac{1}{3})^{\frac{1}{2}}[2^{\frac{1}{2}}R_{0011}\,|n\uparrow) - R_{0010}\,|p\uparrow)] \\ &+ (\tfrac{1}{3})^{\frac{1}{2}}[2^{\frac{1}{2}}R_{1100}\,|p\downarrow) - R_{1000}\,|p\uparrow)] \\ &+ \tfrac{1}{3}[2R_{1111}\,|n\downarrow) - 2^{\frac{1}{2}}R_{1011}\,|n\uparrow) - 2^{\frac{1}{2}}R_{1110}\,|p\downarrow) + R_{1010}\,|p\uparrow)] \end{aligned} \tag{17.7}$$

Since the bare vacuum is invariant under rotations, the matrix elements

$$C_{lt} = (\xi\,|\,R^{\dagger}_{ll_ztt_z}R_{ll_ztt_z}\,|\,\xi) \tag{17.8}$$

are independent of l_z and t_z.¶ The normalization of the physical nucleon state requires

$$C_{00} + C_{01} + C_{10} + C_{11} = 1 \tag{17.9}$$

Furthermore, the symmetry of the theory under exchange of space and isospace requires

$$C_{lt} = C_{tl} \tag{17.10}$$

so that we have the following inequalities:

$$\begin{gathered} 0 < C_{00} < 1 \\ 0 < C_{10} = C_{01} < \tfrac{1}{2} \\ 0 < C_{11} < 1 \end{gathered} \tag{17.11}$$

The constants $\mathfrak{r}_i$ can be readily expressed in terms of the matrices C_{lt}.

¶ This is a general result usually called the Wigner-Eckart theorem. See, e.g., M. E. Rose, "Elementary Theory of Angular Momentum," p. 85, John Wiley & Sons, Inc., New York, 1957.

We have only to remember that the bare nucleon states are eigenstates of the operators $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, e.g.,

$$\tau_3 \mid p) = \sigma_3 \mid p) = \mid p)$$

and that the operators $R_{ll_z tt_z}$ commute with $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. Since the C_{lt} are independent of l_z and t_z, we may calculate any matrix element, e.g.,

$$\begin{aligned} \mathfrak{r}_1 &= \frac{\langle p\uparrow \mid \tau_3 \mid p\uparrow\rangle}{(p\uparrow \mid \tau_3 \mid p\uparrow)} = C_{00} + \tfrac{2}{3}C_{10} - \tfrac{1}{3}C_{11} \\ \mathfrak{r}_2 &= \frac{\langle p\uparrow \mid \sigma_3\tau_3 \mid p\uparrow\rangle}{(p\uparrow \mid \sigma_3\tau_3 \mid p\uparrow)} = C_{00} - \tfrac{2}{3}C_{10} + \tfrac{1}{9}C_{11} \end{aligned} \tag{17.12}$$

The reader may check that the evaluation of, say, $\langle p\downarrow \mid \sigma^{-1} \mid p\uparrow\rangle$ leads to the same result for $\mathfrak{r}_1$. Because of (17.9) and (17.10), only two C's are independent and, hence, can be expressed by the $\mathfrak{r}$'s. By inserting (17.9) and (17.10) into (17.11) and (17.12), we derive several inequalities, e.g.,

$$\begin{aligned} -\tfrac{1}{3} &\le \mathfrak{r}_1 \le 1 & -\tfrac{1}{3} &\le \mathfrak{r}_2 \le 1 \\ -\tfrac{1}{3} &\le 3\mathfrak{r}_2 + 2\mathfrak{r}_1 \le 5 & 1 + 2\mathfrak{r}_1 - 3\mathfrak{r}_2 &\ge 0 \end{aligned} \tag{17.13}$$

There are no further exact statements that can be made. However, there are good reasons to expect C_{10} to be small. Since a single meson has $l = t = 1$, the states in question must have at least two mesons, the angular momenta of which add to 1, while the isospins compensate (or vice versa). Therefore, the angular momentum of the mesonic wave function is odd,[1] whereas the isospin is even under exchange of the mesons (or vice versa), and their Bose-Einstein statistics requires the radial part of the wave function to be odd. This radial antisymmetrization, together with the short-range and exponential radial decay (e.g., $e^{-(k_1r_1+k_2r_2)} - e^{-(k_2r_1+k_1r_2)}$), suggests that the vacuum expectation value which defines C_{01} should be small. This conjecture is strengthened by an exact calculation of such states in an extension of the Lee model which allows the exchanges of more than one meson.[2] In this model $C_{01} < 0.01$. With the assumption that C_{01} is negligible, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are *both* completely determined by a single parameter C_{11} (since $C_{00} = 1 - C_{11}$), and we obtain

$$\begin{aligned} \mathfrak{r}_1 &= 1 - \tfrac{4}{3}C_{11} \\ \mathfrak{r}_2 &= 1 - \tfrac{8}{9}C_{11} = \tfrac{1}{3}(1 + 2\mathfrak{r}_1) \end{aligned} \tag{17.14}$$

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[1] The state with unit total angular momentum of two particles of individual angular momentum 1 is odd under exchange of the two particles (e.g., $\mathbf{k} = \mathbf{k}_1 \times \mathbf{k}_2$). When the angular momenta add to 0, then the state is even under exchange (e.g., $\mathbf{k} = \mathbf{k}_1 \cdot \mathbf{k}_2$).

[2] U. Haber-Schaim and W. Thirring, *Nuovo cimento*, **2**:100 (1955).

The importance of $\mathfrak{r}_1$ and $\mathfrak{r}_2$ is that they are the sole quantities of the physical nucleon states which we shall need to relate the theory to experimental measurements. There have been many attempts, however, to obtain approximate expressions for the operators R as well as for their expectation values. These calculations are illustrations of our general development, and they allow us to calculate $\mathfrak{r}_1$ and $\mathfrak{r}_2$ as well as other matrices in certain limiting, but for the most part unrealistic, cases.[1] In the following we shall outline the main points of various approximations, in current usage, when applied to the ground state.

17.2. Perturbation Theory.[2] In perturbation theory, the coupling constant f and, therefore, H' are considered to be a small disturbance of the free meson field. In an expansion in H' (or f), the first-order change in the ground-state wave function is due to the possible presence of just one meson, which can be emitted by the action of H'. For example, since $H_0 \mid N) = 0$,

$$\begin{aligned} \mid p\uparrow\rangle &= \mid p\uparrow) - \frac{1}{H_0} H' \mid p\uparrow) \\ &= \mid p\uparrow) + i\frac{f}{\mu}\sum_\alpha \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \frac{\boldsymbol{\sigma}\cdot\mathbf{k}\tau_\alpha a_\alpha^\dagger(\mathbf{k})\rho(k)}{(2\omega^3)^{\frac{1}{2}}} \mid p\uparrow) \qquad (17.15) \end{aligned}$$

In this approximation the wave function of the virtual mesons is similar to that of the neutral static-source-model state, aside from the factor **k** and the spin and isospin dependence, which we discussed in the last chapters. It is clear that the wave function is not (but could be) normalized to order f^2. The energy of the ground state being adjusted to zero determines $\mathscr{E}_0$ to order f^2 [provided $(\xi' \mid H' \mid \xi)$ is zero, as it is in the theory considered]:

$$\begin{aligned} \langle p\uparrow \mid H \mid p\uparrow\rangle &= \sum_\alpha (p\uparrow \mid H' \frac{1}{H_0} \omega a_\alpha^\dagger a_\alpha \frac{1}{H_0} H' \mid p\uparrow) \\ &\quad -2(p\uparrow \mid H' \frac{1}{H_0} H' \mid p\uparrow) - \mathscr{E}_0 = 0 \qquad (17.16) \\ \mathscr{E}_0 &= -(p\uparrow \mid \frac{f^2}{\mu^2} \sum_\alpha \int \frac{d^3k}{(2\pi)^3} \frac{\boldsymbol{\sigma}\cdot\mathbf{k}\,\boldsymbol{\sigma}\cdot\mathbf{k}\,\tau_\alpha\tau_\alpha}{2\omega^2} \rho^2(k) \mid p\uparrow) \\ &= -\frac{3}{2}\frac{f^2}{\mu^2} \mathop{\mathrm{S}}_{\mathbf{k}} \frac{k^2}{\omega^2} \rho^2(k) \end{aligned}$$

This energy diverges cubically for a point source, and it is, therefore, very sensitive to the form or cutoff momentum of the source. We can also write (17.15) in the form of (17.7). Only the terms corresponding to C_{00} and C_{11} contribute, because at most one meson is present. Furthermore, $R_{0000} = 1$,

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[1] The reader who is interested only in practical results may omit the remaining sections of this chapter.

[2] See, e.g., R. E. Marshak, "Meson Physics," McGraw-Hill Book Company, Inc., New York, 1952.

$$R_{1010} = (3)^{\frac{1}{2}} \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \tau_3 a_3^{\dagger} \sigma_3 k_3 \frac{\rho(k)}{(2\omega^3)^{\frac{1}{2}}}$$

so that

$$C_{00} = 1 \qquad C_{10} = C_{01} = 0 \qquad C_{11} = \frac{3}{2}\frac{f^2}{\mu^2} \mathop{\mathrm{S}}_{\mathbf{k}} \frac{k^2}{\omega^3} \,|\, \rho(k) \,|^2 \tag{17.17}$$

Because the state $|\, p\uparrow \rangle$ is not normalized, (17.9) and (17.13) do not apply; from (17.12), however, we find

$$\mathfrak{r}_1 = 1 - \frac{f^2}{2\mu^2} \mathop{\mathrm{S}}_{\mathbf{k}} \frac{k^2}{\omega^3} |\rho(k)|^2 \tag{17.18}$$

$$\mathfrak{r}_2 = 1 + \frac{1}{6}\frac{f^2}{\mu^2} \mathop{\mathrm{S}}_{\mathbf{k}} \frac{k^2}{\omega^3} |\rho(k)|^2 \tag{17.19}$$

If $f^2\,(k_{\max}/\mu)^2 > 1$, then the inequalities (17.13) cannot even be satisfied approximately. This is due to the fact that the state function is not normalized to order f^2. The perturbation approach is clearly valid only if $C_{11} \ll 1$, and the whole expansion in powers of f (e.g., $fk_{\max}/\mu$) becomes impracticable unless this is the case. For example, for $k_{\max} \sim M$, this limits f to $f \lesssim 0.1$ and $f^2/4\pi \lesssim 10^{-3}$, whereas we shall see later that experiments require $f^2/4\pi \sim 10^{-1}$.

17.3. Tamm-Dancoff Approximation.[1] The Tamm-Dancoff method attempts to remedy some of the shortcomings of perturbation theory. It is similar in that it limits the number of mesons in the cloud (usually to one) but differs in that it solves the Schrödinger equation in this subspace. Correspondingly, it agrees with the exact solution of the Lee model for the $Q = \frac{1}{2}$ states. From (17.7), the general form of the ground state in the one-meson approximation considered is

$$|\, p\uparrow \rangle = \mathcal{N}\left[1 + \sum_{\alpha} \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} a_{\alpha}^{\dagger}(\mathbf{k}) \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{k} \tau_{\alpha} f(\mathbf{k})\right] |\, p\uparrow) \tag{17.20}$$

where $\mathcal{N}$ is a normalization constant,

$$\mathcal{N}^{-2} = 1 + 3 \mathop{\mathrm{S}}_{\mathbf{k}} |f(\mathbf{k})|^2 \tag{17.21}$$

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[1] I. Tamm, *J. Phys.* (*U.S.S.R.*), **9**:449 (1945); S. M. Dancoff, *Phys. Rev.*, **78**:382 (1950). See also H. A. Bethe and F. de Hoffmann, "Mesons and Fields," vol. II, Row, Peterson & Company, Evanston, Ill., 1955.

and $f(\mathbf{k})$ describes the meson momentum-space wave function. When (17.20) is substituted into the Schrödinger equation[1]

$$H \mid p\uparrow\rangle = (H_0 + H' - \mathscr{E}_0) \mid p\uparrow\rangle = 0 \tag{17.22}$$

then the application of H' to the term proportional to $a^\dagger(\mathbf{k}) \mid p)$ creates states with two and no mesons. In the spirit of the approximation stated earlier, the two-meson amplitude is neglected, and we obtain

$$\begin{aligned}\mathscr{N}\biggl[-\mathscr{E}_0 - i\frac{f}{\mu}\sum_\alpha \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\boldsymbol{\sigma}\cdot\mathbf{k}a_\alpha^\dagger(\mathbf{k})\rho(k)}{(2\omega)^{1/2}}\tau_\alpha \\ + \sum_\alpha \int a_\alpha^\dagger(\mathbf{k})\frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{k}\tau_\alpha f(\mathbf{k})(\omega - \mathscr{E}_0)\frac{d^3k}{(2\pi)^{3/2}} \\ + i\frac{3f}{\mu}\int \frac{d^3k}{(2\pi)^3}\frac{f(\mathbf{k})k\rho(k)}{(2\omega)^{1/2}}\biggr] \mid p\uparrow) = 0 \end{aligned} \tag{17.23}$$

We deduce, therefore,

$$f(\mathbf{k}) = i\frac{f}{\mu}\frac{k}{(2\omega)^{1/2}}\frac{\rho(k)}{\omega - \mathscr{E}_0} \tag{17.24}$$

and

$$\mathscr{E}_0 = -\frac{3f^2}{\mu^2}\mathop{\mathrm{S}}_{\mathbf{k}}\frac{k^2\rho^2(k)}{2\omega(\omega - \mathscr{E}_0)} \tag{17.25}$$

In the continuum limit, e.g., $\mathop{\mathrm{S}}_{\mathbf{k}} = d^3k/(2\pi)^3$, (17.25) is an integral equation for the number $\mathscr{E}_0$. The wave function $f(\mathbf{k})$, as anticipated, is analogous to that of the Lee model. In the weak-coupling limit

$$\frac{3}{2}\frac{f^2}{\mu^2}\mathop{\mathrm{S}}_{\mathbf{k}}\frac{k^2\,|\rho(k)|^2}{\omega^2} \ll k_{\max}$$

and $f(\mathbf{k})$ and $\mathscr{E}_0$ go over into the perturbation-theory result. On the other hand, for the more realistic limit

$$\frac{3}{2}\frac{f^2}{\mu^2}\mathop{\mathrm{S}}_{\mathbf{k}}\frac{k^2\,|\rho(k)|^2}{\omega} \gg k_{\max}^2$$

we obtain (using the same sign as in perturbation theory)

$$\mathscr{E}_0 = -\left[3\frac{f^2}{\mu^2}\mathop{\mathrm{S}}_{\mathbf{k}}\frac{k^2\,|\rho(k)|^2}{2\omega}\right]^{1/2} \tag{17.26}$$

Whereas for small f^2 the quantities C_{11}, C_{01}, and C_{00} correspond to those of perturbation theory, in the opposite limit considered above we find

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[1] This can also be considered to be a variational procedure with the trial state (17.20). This method will be developed in the next section.

$$f(\mathbf{k}) = -i\frac{f}{\mu}\frac{k}{(2\omega)^{\frac{1}{2}}}\frac{\rho(k)}{\mathscr{E}_0} \qquad \text{and} \qquad \mathscr{N} = (\tfrac{1}{2})^{\frac{1}{2}}$$

so that

$$\begin{aligned} &C_{00} = \tfrac{1}{2} \qquad C_{01} = C_{10} = 0 \qquad C_{11} = \tfrac{1}{\text{[illegible]}} \\ &\mathfrak{r}_1 = \tfrac{1}{3} \qquad \mathfrak{r}_2 = \tfrac{5}{9} \end{aligned} \tag{17.27}$$

in fair agreement with the experimental numbers that we shall find later.

Since the physical-nucleon-state function is normalized, the inequalities (17.13) are satisfied in this limiting case. It is not clear, however, whether the results resemble the exact solution. There is no a priori reason why two-meson and higher amplitudes should be negligible for large $fk_{\max}/\mu$.

17.4. Tomonaga Intermediate-coupling Approximation.[1] The Tomonaga method for intermediate coupling strengths does not limit the number of mesons in the cloud. Instead, it takes advantage of the Bose-Einstein statistics of the pions, which favors a clustering and assumes that they all have the same radial wave functions. For this reason, it is simpler to go to the angular-momentum representation, Eq. (5.10*a*).[2] Thus, if we expand the meson field in a complete set of orthonormal radial functions $F_s(k)$ in the sense that

$$\begin{gathered} a_\alpha(\mathbf{k}) = \sum_{s,l,m} \frac{1}{k} F_s(k) Y_l^m(\theta_k,\varphi_k) a_{\alpha s l}^m \\ \sum_s F_s^*(k')F_s(k) = \delta(k' - k) \qquad \int_0^\infty dk\, F_s^*(k)F_{s'}(k) = \delta_{ss'} \end{gathered} \tag{17.28}$$

then the approximation for the ground state consists in assuming a trial function (to be determined by a variational principle) that involves only one mode, say $s = 0$. The new operators $a_{\alpha s l}^m$ obey the usual commutation relations for every value of s:

$$[a_{\alpha s l}^m, a_{\alpha' s' l'}^{\dagger m'}] = \delta_{\alpha\alpha'}\delta_{ss'}\delta_{ll'}\delta_{mm'} \tag{17.29}$$

The Hamiltonian is then quite generally

$$\begin{aligned} H_0 &= \sum_\alpha \int d^3k\, a_\alpha^\dagger(\mathbf{k})a_\alpha(\mathbf{k})\omega = \sum_{\alpha,s,l,m,s'} a_{\alpha s l m}^\dagger a_{\alpha s' l m} W_{ss'} \\ H' &= \sum_{\alpha,s,l,m} \tau_\alpha \sigma_m (f_s a_{\alpha s l m} + f_s^* a_{\alpha s l m}^\dagger) \end{aligned} \tag{17.30}$$

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[1] S. Tomonaga, *Progr. Theoret. Phys.* (*Kyoto*), **2**:6 (1947). See also T. D. Lee and D. Pines, *Phys. Rev.*, **92**:883 (1953); T. D. Lee and R. Christian, *Phys. Rev.*, **94**:1760 (1954); M. H. Friedman, T. D. Lee, and R. Christian, *Phys. Rev.*, **100**:1494 (1955); E. M. Henley and T. D. Lee, *Phys. Rev.*, **101**:1536 (1956).

[2] This could also have been done in the last two sections.

with
$$W_{ss'} = \int_0^\infty dk\, \omega F_s^*(k) F_{s'}(k) \qquad \bar{f}_s = \frac{f}{\mu} \int_0^\infty dk \frac{\rho(k) k^2 F_s(k)}{(12\pi^2\omega)^{\frac{1}{2}}} \tag{17.31}$$

In the Tomonaga approximation only the term $s = 0$ is retained[1] by taking a trial state $| N_t\rangle$ with $a_s | N_t\rangle = 0$ for $s \neq 0$. The best form of $F_0(k) \equiv F(k)$ is found by minimizing

$$\mathscr{E}_0 = \langle N_t | H_0 + H' | N_t\rangle \tag{17.32}$$

with respect to the form of $F(k)$ and the dependence of $| N_t\rangle$ on the meson-creation and -destruction operators $a_{\alpha j}$ and $a_{\alpha j}^\dagger$. In the former procedure, we take into account the restraint

$$\int_0^\infty dk\, |F(k)|^2 = 1$$

by introducing a Lagrangian multiplier λ'. We obtain

$$\begin{aligned}\frac{\delta \mathscr{E}_0}{\delta F^*(k)} &= \langle N_t | \sum_{\alpha j} a_{\alpha j}^\dagger a_{\alpha j} | N_t\rangle \frac{\delta W}{\delta F^*(k)} \\ &\quad + \langle N_t | \sum_{\alpha j} \tau_\alpha \sigma_j a_{\alpha j}^\dagger | N_t\rangle \frac{\delta \bar{f}^*}{\delta F^*(k)} + \lambda' F(k) \\ &= \langle N_t | \sum_{\alpha j} a_{\alpha j}^\dagger a_{\alpha j} | N_t\rangle \omega F(k) \\ &\quad + \langle N_t | \sum_{\alpha j} \tau_\alpha \sigma_j a_{\alpha j}^\dagger | N_t\rangle \frac{k^2 f \rho(k)}{\mu (12\omega)^{\frac{1}{2}} \pi} + \lambda' F(k) = 0\end{aligned}$$

and thus find, for the form of $F(k)$,

$$F(k) = \frac{\mathscr{N} k^2 \rho(k)}{\omega^{\frac{1}{2}}(\omega + \lambda)} \tag{17.33}$$

where $\mathscr{N}$ is a normalization constant independent of k and where λ is an undetermined multiplier, related to λ' by

$$\lambda = \frac{\lambda'}{\langle N_t | \sum_{\alpha j} a_{\alpha j}^\dagger a_{\alpha j} | N_t\rangle}$$

Minimizing $\mathscr{E}_0$ with this form of $F(k)$, we get the lowest eigenvalue as a function of λ. This parameter is then to be determined by a further variational calculation, which will be carried out in the next section. The form of $F(k)$ determined above is the same as that of the Tamm-Dancoff approximation (17.24), but it should be remembered that the number of mesons is not limited.

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[1] Since only $l = 1$ and $s = 0$ enter into the following, we shall henceforth drop the subscripts l and s from $a_{\alpha s l m}$.

It remains to determine the ground-state eigenfunction $| N_t \rangle$ and the energy $\mathscr{E}_0$, which depends on W and $\bar{f}$, and, therefore, on λ. The reduced Hamiltonian, (17.30) with $s = 0$ only, corresponds to nine oscillators ($\alpha = 1, 2, 3$; $j = 1, 2, 3$) coupled to a spin and an isospin, and its diagonalization is a problem in elementary wave mechanics. However, it cannot be carried out in closed form. This would be true even if we had the complications due only to spin (neutral pseudoscalar theory) or to isospin (symmetric scalar theory). To gain some insight into the problem, we shall consider the former case[1] in some detail. Here the reduced Hamiltonian is[2]

$$H = W\sum_j a_j^\dagger a_j + \bar{g}\sigma_j(a_j + a_j^\dagger) - \mathscr{E}_0 \tag{17.34}$$

Going back to the canonically conjugate operators $\mathbf{p}$ and $\mathbf{q}$ by

$$\begin{aligned} p_j &= -i(a_j - a_j^\dagger)\left(\frac{W}{2}\right)^{\frac{1}{2}} \\ q_j &= (a_j + a_j^\dagger)\left(\frac{1}{2W}\right)^{\frac{1}{2}} \\ [q_j,p_{j'}] &= i\,\delta_{jj'} \end{aligned} \tag{17.35}$$

we can rewrite H in the form

$$H = \tfrac{1}{2}(\mathbf{p}^2 + W^2\mathbf{q}^2) + g'\boldsymbol{\sigma}\cdot\mathbf{q} - \tfrac{3}{2}W - \mathscr{E}_0 \tag{17.36}$$

with $\quad \mathbf{q}^2 = \sum_{j=1}^{3} q_j^2 \quad$ and $\quad g' = \bar{g}(2W)^{\frac{1}{2}}$

The form (17.36) emphasizes the formal analogy to elementary wave mechanics and corresponds to a three-dimensional oscillator coupled to a spin by means of an interaction $\boldsymbol{\sigma}\cdot\mathbf{q}$. This interaction does not conserve parity, since $\boldsymbol{\sigma} \to \boldsymbol{\sigma}$, but $\mathbf{q} \to -\mathbf{q}$ under a reflection in the origin. By carrying the analogy further, we note that the ground state should be a mixture of an angular momentum $S_{\frac{1}{2}}$ and a $P_{\frac{1}{2}}$ state.[3] These are the only states allowed by the restriction $j = \frac{1}{2}$ for the bare and physical nucleon, and they correspond to angular-momentum $l = 0$ and $l = 1$ for the meson cloud. These states will be of the form[4] $h_0(q) \mid N \rangle$ and $\boldsymbol{\sigma}\cdot\mathbf{q}h_1(q) \mid N \rangle$. However, we shall not expand in terms of these states but rather in terms of those for which $\boldsymbol{\sigma}$ is parallel or antiparallel to $\mathbf{q}$, since this determines the sign of H'. Introducing the projection operators into the eigenstates of $\boldsymbol{\sigma} \cdot \mathbf{q}$,

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[1] To distinguish this case, we shall replace f by $\bar{g}$, as was done in Chap. 16.

[2] We assume that λ is real; therefore, $\bar{g} = \bar{g}^*$, and $F(k)$ as determined by (17.33) is then real, too.

[3] The theorem of level ordering in a central potential is applicable and proves that the ground state actually is a $j = \frac{1}{2}$ state. See R. G. Sachs, "Nuclear Theory," appendix 1, Addison-Wesley Publishing Company, Reading, Mass., 1953.

[4] $q \equiv |\mathbf{q}|$.

$$\mathfrak{P}_{\pm} = \frac{1}{2}\left(1 \pm \frac{\boldsymbol{\sigma} \cdot \mathbf{q}}{q}\right) \tag{17.37}$$

we write the ground state as

$$| N\rangle = \left[\mathfrak{P}_{+} \frac{h_{+}(q)}{q} + \mathfrak{P}_{-} \frac{h_{-}(q)}{q}\right] | N) \tag{17.38}$$

where $h_{\pm}$ satisfies

$$2\pi\int_{0}^{\infty} dq\, [|h_{+}(q)|^{2} + |h_{-}(q)|^{2}] = 1 \qquad h_{+}(0) = h_{-}(0) = 0 \tag{17.39}$$

Since $\mathfrak{P}_{\pm}$ is rotation-invariant, (17.38) is an eigenstate of $\mathbf{J}$ with eigenvalue $\frac{1}{2}$ and of J_z with eigenvalue $\pm\frac{1}{2}$, depending on whether $| N)$ has spin up or down. Evaluating $H \,| N\rangle = 0$ with the aid of

$$\boldsymbol{\sigma} \cdot \mathbf{A}\, \boldsymbol{\sigma} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot \mathbf{A} \times \mathbf{B}$$

$$p^{2} = -\frac{1}{q^{2}}\frac{d}{dq}\left(q^{2}\frac{d}{dq}\right) + \frac{l(l+1)}{q^{2}}$$

we find

$$\left[-\frac{1}{2}\frac{d^{2}}{dq^{2}} + \frac{1}{2}W^{2}\left(q \pm \frac{g'}{W^{2}}\right)^{2} - \left(\mathscr{E}_{0} + \frac{g'^{2}}{2W^{2}} + \frac{3}{2}W\right)\right]h_{\pm} = \pm\frac{h_{\mp}}{q^{2}} \tag{17.40}$$

These two second-order equations cannot be solved analytically but can be readily discussed in limiting cases.

Whereas for small values of g' we are led back to the perturbation result (e.g., h_0 dominant), for large $g' > 0$ we find that h_{-} becomes the leading term. This can be seen by realizing that (17.40) corresponds to a harmonic oscillator displaced by $\pm g'/W^{2}$ from the center (Fig. 17.1). Since we are concerned only with $q > 0$, h_{-} will be a gaussian function around g'/W^{2} with the same width as the ground state, whereas h_{+} has to be kept as small as possible. Indeed, for large displacements the coupling term h_{-}/q^{2} will become negligible, and a pure h_{-} solution will be a good approximation. Furthermore, if the displacement is much larger than the zero-point fluctuation $\sim W^{-\frac{1}{2}}$, only a small adjustment of h_{-} will be needed to meet the boundary condition at $q = 0$. Hence, in this limit the ground state will be approximately

$$| N\rangle = \frac{1}{9}\left(\frac{W}{16\pi^{3}}\right)^{\frac{1}{4}}\left(1 - \frac{\boldsymbol{\sigma} \cdot \mathbf{q}}{q}\right) \exp\left[-\frac{W}{2}\left(q - \frac{g'}{W^{2}}\right)^{2}\right] | N) \tag{17.41}$$

with

$$\mathscr{E}_{0} = -\frac{g'^{2}}{2W^{2}} - W \tag{17.42}$$

The other solution of (17.40), which is predominantly h_+, will have a much higher energy. For small g' the two solutions will go over into the $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ level, respectively. Similarly, the energy of one of the $\frac{3}{2}$-states will be lowered by turning on H'. Indeed, a perturbation treatment of the centrifugal term shows that it will be above the ground state by

$$\Delta E = l(l+1)\left\langle \frac{1}{2q^2} \right\rangle \sim \frac{W^4}{g'^2}$$

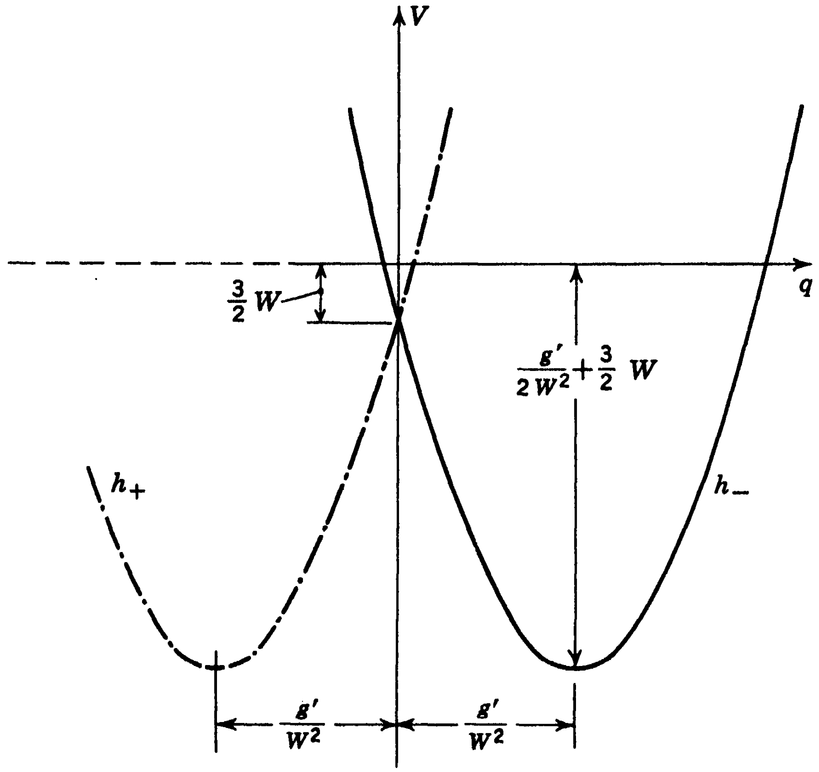

Fig. 17.1. Plot of the potential $V = \frac{1}{2}W^2\left(q \pm \frac{g'}{W^2}\right) - \left(\frac{g'}{2W^2} + \frac{3}{2}W\right)$ for h_+ and h_-. The curves have been continued into the unphysical region of negative values of q, where they are shown by dashed lines.

With increasing g' the energy approaches the ground state, as is depicted in Fig. 17.2. Of course, whether this excited state will be stable or will decay into the ground state with pion emission can be decided only by taking into account the continuum meson modes ($s \neq 0$) in (17.30). The problem will be similar to the Lee model, where there is a source with various energy levels coupled to the meson field. If the self-energy shifts of all levels due to this coupling are the same, then the first excited state will be stable if its energy is less than μ above the ground state.

The excited $\frac{3}{2}$-state corresponds classically to the spin gyration we studied earlier. Indeed, since for strong coupling we find $\lambda \rightarrow 0$ and so

$$F(k) \cong \frac{2^{\frac{1}{2}}k^2\rho(k)}{k_{\max}\omega^{\frac{3}{2}}} \qquad W_{00} \cong \frac{2}{3}k_{\max} \qquad g' = \frac{g}{\mu}\frac{k_{\max}^2}{9\pi}(2k_{\max})^{\frac{1}{2}}$$

we find for the excitation energy estimated above

$$\Delta E = \frac{W^4}{g'^2} = \frac{2\pi}{k_{\max}(g^2/4\pi\mu^2)}$$

in fairly close agreement with the classical result from (16.18),

$$\Delta E = \frac{9\pi}{4k_{\max}\,g^2/4\pi\mu^2}$$

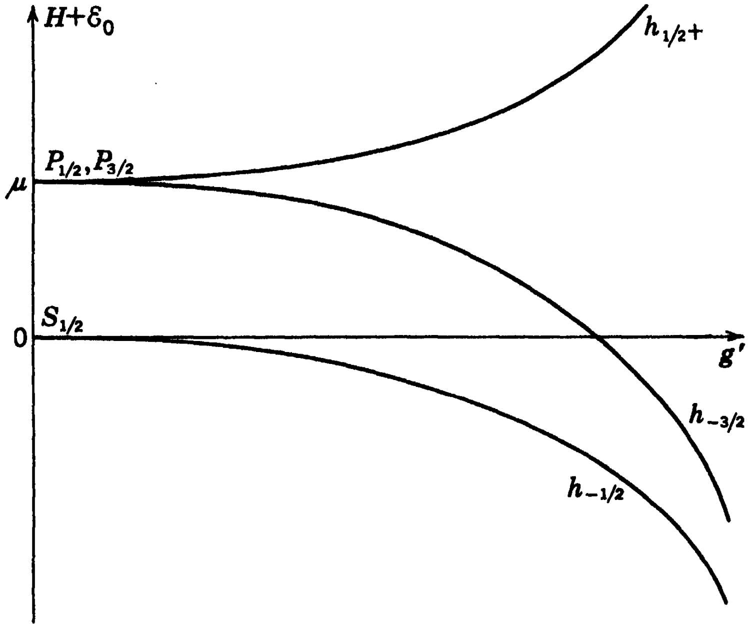

Fig. 17.2. Plot of the energy $H + \mathcal{E}_0$ for the lowest states as a function of the effective coupling strength g'. The states are labeled by their angular momenta and other appropriate quantum numbers.

To mention some pertinent features of the ground state, we first remark that the gaussian form of (17.41) implies a Poisson-like distribution of virtual mesons, as in the neutral scalar theory. However, q only creates mesons with total angular momentum 0 and, therefore, a mixture of meson pairs. Indeed, if

$$q_x = x \qquad q_y = y \qquad q_z = z$$

then

$$q^2 = (x + iy)(x - iy) + z^2 = q^2(Y_1^{1*}Y_1^1 + Y_1^{-1*}Y_1^{-1} + Y_1^{0^2})\frac{4\pi}{3}$$

$$= q^2(-2Y_1^{-1}Y_1^1 + Y_1^{0^2})\frac{4\pi}{3}$$

shows that $h_0(q)$ and $h_1(q)$ always create a mixture of pairs with $l_z = \pm 1$ or with $l_z = 0$, the former possibility having twice the amplitude of the latter. In the neutral pseudoscalar theory, the constant $\mathfrak{r}_2$ does not exist, and in the limit of very strong coupling, $\mathfrak{r}_1$ is determined by

$$\begin{aligned}\langle N_t \,|\, \boldsymbol{\sigma} \,|\, N_t\rangle &= \mathcal{N}^2(N \,|\, \int d^3q \exp\left[-W\left(q - \frac{g'}{W^2}\right)^2\right] \\ &\qquad\qquad \times \frac{1}{q^2}\left(1 - \frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{q}\right)\boldsymbol{\sigma}\left(1 - \frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{q}\right) \,|\, N) \\ &= \tfrac{1}{3}(N \,|\, \boldsymbol{\sigma} \,|\, N) \qquad (17.43)\end{aligned}$$

Hence, $\mathfrak{r}_1$ in this theory and in the limit considered is $\frac{1}{3}$. For intermediate values of g' the system of equations can be handled only by numerical methods, which we shall not discuss here.[1]

Returning to the symmetric pseudoscalar theory, we can easily imagine that an analytic solution is not feasible. Because of the extra degrees of freedom brought in by the isospin, we obtain, instead of (17.39), four simultaneous equations for four functions $h_0, \ldots, h_3$ in a nine-dimensional space. However, for small coupling strengths f the perturbation result is obtained, whereas for large values of f the theory goes over into the strong-coupling approximation, which we shall discuss next.

17.5. Strong-coupling Approximation.[2] In this limit many mesons surround the bare nucleon, and the zero-point fluctuations are much smaller than the mean value of the field, so that classical calculations begin to acquire meaning. In the simple treatment here, where our main interest is in the ground-state properties, all virtual mesons are put into the same mode, and the resulting reduced Hamiltonian (17.30) is diagonalized in the limit of large $\bar{g}$. If we introduce canonical operators $p_{\alpha j}$ and $q_{\alpha j}$, as in (17.35),

$$p_{\alpha j} = -i(a_{\alpha j} - a_{\alpha j}^\dagger)\left(\frac{W}{2}\right)^{\frac{1}{2}}$$

$$q_{\alpha j} = (a_{\alpha j} + a_{\alpha j}^\dagger)\left(\frac{1}{2W}\right)^{\frac{1}{2}} \qquad (17.44)$$

$$[q_{\alpha j}, p_{\alpha' j'}] = i\delta_{\alpha\alpha'}\delta_{jj'}$$

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[1] See, however, references listed in footnote 1, page 186.

[2] W. Pauli and S. M. Dancoff, *Phys. Rev.*, **62**:851 (1942). See also G. Wentzel, *Helv. Phys. Acta*, **13**:269 (1940) and **14**, 633 (1941); R. Serber and S. M. Dancoff, *Phys. Rev.*, **62**:85 (1942); F. Harlow and B. A. Jacobsohn, *Phys. Rev.*, **93**:333 (1954); A. Pais and R. Serber, *Phys. Rev.*, **105**:1636 (1959) and **113**:955 (1959).

then the reduced Hamiltonian becomes

$$\begin{aligned} H^r &= H_0^r + H'^r - \mathscr{E}_0 \\ &= \sum_{\alpha j} \frac{1}{2}(p_{\alpha j}^2 + W^2 q_{\alpha j}^2) + f'\sigma_j \tau_\alpha q_{\alpha j} - \frac{9}{2} W - \mathscr{E}_0 \end{aligned} \tag{17.45}$$

where

$$f' = \bar{f}(2W)^{\frac{1}{2}}$$

To find the eigenvalues of H, we follow the pattern of the last section and, for the moment, resort to a semiclassical treatment in which the operators q are considered to be ordinary numbers and the minimum of $H(q, p = 0)$ is determined. We must then find the lowest eigenvalue of the 4×4 matrix $\tau_\alpha \sigma_j q_{\alpha j}$. To this end, we note that a rotation in both spin and isospin space[1]

$$\begin{aligned} U\tau_\alpha U^{-1} &= \tau_\beta A_{\beta\alpha} \qquad & A^* &= A \qquad & AA^\dagger &= 1 \\ U\sigma_j U^{-1} &= \sigma_i B_{ij} \qquad & B^* &= B \qquad & BB^\dagger &= 1 \end{aligned}$$

induces the transformation

$$U\tau_\alpha \sigma_j q_{\alpha j} U^{-1} = \tau_\alpha \sigma_j Q_{\alpha j} \tag{17.46}$$

where (in shorthand notation)

$$Q_{\alpha j} = A_{\alpha\beta} B_{jk} q_{\beta k} = AqB^{-1} \tag{17.47}$$

We claim that q can be diagonalized by this transformation and demonstrate it as follows. We observe, first of all, that the real positive definite symmetric matrix[2] qq^T can be diagonalized by the real orthogonal matrix A:

$$QQ^T = Aqq^T A^{-1} \tag{17.48}$$

We can, therefore, extract the square root to obtain the real diagonal matrix Q with elements $Q_{\alpha j} = Q_\alpha \delta_{\alpha j}$. The elements Q_α are the analogue of the radial variable q in the neutral theory and will, therefore, be assumed positive. The matrix B can be written by means of (17.47) as

$$B = Q^{-1} Aq \tag{17.49}$$

and can readily be shown to be orthogonal ($Q^T = Q$):

$$BB^T = Q^{-1} Q^2 Q^{-1} = 1 \tag{17.50}$$

The eigenvalues of H' which are proportional to those of the 4×4 matrix,

$$\tau_\alpha \sigma_j Q_{\alpha j} = \sum_{i=1}^{3} Q_i \tau_i \sigma_i \tag{17.51}$$

[1] The properties of the rotation matrices are determined by the hermiticity of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. A sum over subscripts appearing twice is to be understood.

[2] q^T = transpose matrix of q.

can be found by making use of the properties of $\mathcal{O}_i = \sigma_i\tau_i$, namely,

$$\mathcal{O}_i^2 = 1 \qquad \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k = -1 \qquad [\mathcal{O}_i,\mathcal{O}_j] = 0$$

Hence, the operators $\mathcal{O}_i$ can be simultaneously diagonalized. The eigenvalues of $\mathcal{O}_i$ are ± 1, and the product of a set of eigenvalues of $\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k$ must be -1. Therefore, the four possible eigenvalues of (17.51) are

$$\begin{matrix} -Q_1 - Q_2 - Q_3 & \quad Q_1 + Q_2 - Q_3 \\ Q_1 - Q_2 + Q_3 & \quad -Q_1 + Q_2 + Q_3 \end{matrix} \tag{17.52}$$

Because of our assumption that $Q_i > 0$, the first of these corresponds to the ground state.

To find the minimum energy of the ground state and the best form of Q_i, we minimize the particular form of the potential energy of H in the ground state,

$$\begin{aligned} \tilde{H}^r &= \sum_{\alpha j}\left[\frac{1}{2}W^2q_{\alpha j}^2 + f'(\sigma_j\tau_\alpha)_{00}q_{\alpha j}\right] \\ &= \sum_i \frac{1}{2}W^2Q_i^2 - f'Q_i \end{aligned} \tag{17.53}$$

with respect to Q_i. The minimum is reached at

$$Q_i = \frac{f'}{W^2} \tag{17.54}$$

and is

$$\tilde{H}^r = \frac{3}{2}\frac{f'^2}{W^2} - 3\frac{f'^2}{W^2} = -\frac{3}{2}\frac{f'^2}{W^2} \tag{17.55}$$

The ground state of H will contain a gaussian function of the radial variables Q_i centered about their equilibrium values f'/W^2, e.g.,

$$\prod_i \exp\left[-\frac{W}{2}\left(Q_i - \frac{f'}{W^2}\right)^2\right]$$

However, to get a form like (17.41), we still need a projection operator into the eigenstate of $q_{\alpha j}\tau_\alpha\sigma_j$ with eigenvalue $-\sum_i Q_i$. A simple form for this operator can be obtained only at the equilibrium position and will be useful, inasmuch as the widths of the gaussian functions are negligible compared with their displacement. For $Q_i = f'/W^2$ we have

$$q_{\alpha j} = (A^{-1}QB)_{\alpha j} = \frac{f'}{W^2}A_{\alpha\beta}^{-1}B_{\beta j} = \frac{f'}{W^2}e_{\alpha j} \tag{17.56}$$

where the e satisfy

$$e_{\alpha j}e_{\alpha k} = \delta_{jk} \qquad e_{\alpha j}e_{\beta j} = \delta_{\alpha\beta}$$

$$e_{\alpha j}e_{\beta k}\epsilon_{\alpha\beta\gamma} = \epsilon_{jkl}e_{\gamma l}$$

Since at the equilibrium

$$(q_{\alpha j}\tau_\alpha\sigma_j)^2 = \frac{3f'^2}{W^4} - \frac{2f'}{W^2} q_{\alpha j}\tau_\alpha\sigma_j$$

we recognize that

$$\mathfrak{P} = \frac{1}{4}\left(1 - q_{\alpha j}\tau_\alpha\sigma_j \frac{W^2}{f'}\right) \tag{17.57}$$

is the desired projection operator. In fact, for $Q_i = f'/W^2$ we have

$$\mathfrak{P}^2 = \mathfrak{P} \qquad q_{\alpha j}\tau_\alpha\sigma_j \mathfrak{P} = -3\frac{f'}{W^2}\mathfrak{P}$$

Thus, in this limit the ground state is approximately

$$| N\rangle = \mathcal{N}(1 - q_{\alpha j}\tau_\alpha\sigma_j W^2 f'^{-1}) \exp\left[-\frac{W}{2}\sum_i \left(Q_i - \frac{f'}{W^2}\right)^2\right] | N) \tag{17.58}$$

Since Q_i and $\mathfrak{P}$ are invariant under rotations in spin and isospin space, the ground state (17.58) is an eigenstate of **J** and **T** with the same eigenvalue as $| N)$. The term proportional to $q_{\alpha j}\tau_\alpha\sigma_j$ is the $l = t = 1$ component of the meson cloud, and the other term corresponds to $l = t = 0$. There are no mixed terms, and by means of (17.8) and (17.12) we obtain

$$\begin{gathered} C_{00} = \tfrac{1}{4} \qquad C_{11} = \tfrac{3}{4} \qquad C_{10} = 0 \\ \mathfrak{r}_1 = 0 \qquad \mathfrak{r}_2 = \tfrac{1}{3} \end{gathered} \tag{17.59}$$

That $\mathfrak{r}_1 = 0$ means that the expectation values for the ground state of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ vanish. This can be verified directly, since

$$\langle N | \tau_\alpha | N\rangle = \frac{1}{4}(N | \tau_\alpha + \sum_{\beta=1}^{3} \tau_\beta\tau_\alpha\tau_\beta | N) = 0$$

Since the ground-state energy is zero, we obtain

$$\mathscr{E}_0 = -\frac{3}{2}\frac{f'^2}{W^2} + E' \tag{17.60}$$

where E' is the sum of the zero-point energy, $-\frac{9}{2}W$, and the kinetic-energy term $\Sigma_{j\alpha}\frac{1}{2}p^2_{\alpha j}$, which is of the order of $\frac{3}{2}W$. For large values of f' we can neglect E' in the ground state and find, by means of (17.31) and (17.33),

$$\begin{aligned} \mathscr{E}_0 = -3\frac{f^2}{W} &= -\frac{f^2}{\mu^2} \frac{\left[\int_0^\infty dk \frac{\rho^2(k)k^4}{\omega(\omega+\lambda)}\right]^2}{\int_0^\infty dk \frac{k^4\rho^2(k)}{(\omega+\lambda)^2}} \frac{1}{4\pi^2} \\ &= -\frac{f^2}{\mu^2}\left(\frac{1}{2\pi}\right)^2 \frac{L_1^2(\lambda)}{L_1(\lambda) - \lambda L_2(\lambda)} \end{aligned} \tag{17.61}$$

with
$$L_n(\lambda) = \int_0^\infty dk\, \frac{k^4\rho^2(k)}{\omega(\omega+\lambda)^n} \tag{17.62}$$

To find the best value of λ, we minimize $\mathscr{E}_0$ with respect to it. Making use of

$$\frac{\partial L_n}{\partial \lambda} = -nL_{n+1}$$

we find

$$\frac{\partial \mathscr{E}_0}{\partial \lambda} = -2\frac{f^2}{\mu^2}\lambda\, \frac{L_2^2 - L_1L_3}{(L_1-\lambda L_2)^2}\, L_1\left(\frac{1}{2\pi}\right)^2$$

The minimum value of $\mathscr{E}_0$ is therefore obtained for $\lambda = 0$ and is

$$\mathscr{E}_0 = -\frac{f^2}{\mu^2}\left(\frac{1}{2\pi}\right)^2 \int_0^\infty dk\, \frac{k^4\rho^2(k)}{\omega^2} \tag{17.63}$$

Hence, in the strong-coupling limit, the self-energy[1,2] is one-third of that in perturbation theory.

As for the neutral pseudoscalar theory, the first excited state will be one with higher spin and isospin, whereas the states corresponding to the other eigenvalues of $\sigma_j\tau_\alpha q_{\alpha j}$ lie much higher. The next level is found to be a $\frac{3}{2}$, $\frac{3}{2}$-level ($J = T = \frac{3}{2}$) and is at the energy

$$\Delta E = \frac{9\pi}{4k_{\max}(f^2/4\pi\mu^2)} \tag{17.64}$$

above the ground state. If this state is unstable, it will produce a resonance in the $\frac{3}{2}$,$\frac{3}{2}$-scattering. We have already studied this in the classical approximation, and we shall study its quantum-mechanical aspects in the next chapter. The results we have obtained here are not exact, even if $f \gg 1$, since all mesons were assumed to be in only one mode and it was not shown that the other modes can be neglected.

17.6. Numerical Methods. The ground-state problem has also been attacked by more elaborate variational methods, which require considerable numerical work.[3] From these investigations we have learned the following. For small values of f^2 and $k_{\max}$, the constants $\mathfrak{r}_1$ and $\mathfrak{r}_2$ vary rather rapidly from their weak-coupling limits 1, 1 to the strong-coupling limits 0, $\frac{1}{3}$. Once the latter are attained, the situation remains

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[1] From (17.16) we have

$$\mathscr{E}_0 = -\frac{3}{(2\pi)^2}\frac{f^2}{\mu^2}\int_0^\infty dk\, \frac{k^4}{\omega^2}\rho^2(k)$$

[2] See Pauli and Dancoff, *op. cit.*

[3] G. Eder, *Nuovo cimento*, **18**:430 (1960); F. R. Halpern, *Phys. Rev.*, **107**:1145 (1957); F. R. Halpern, L. Sartori, K. Nishimura, and R. Spitzer, *Ann. Phys.*, **7**:154 (1959).

rather insensitive to f^2 and $k_{\max}$. Empirically, we shall see that the best values are[1] $f^2/4\pi \sim 0.2$ and $k_{\max} \sim 5\mu$. The variational calculations indicate that with these values we have not quite reached the strong-coupling limit, and actually $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are experimentally determined to be $\frac{1}{3}$ and $\frac{1}{2}$. Thus, it is quite conceivable that these values correspond to the exact result of the theory. However, such an agreement would not be very significant, since we are still in the region where numerical values depend sensitively on the (unrenormalized) coupling constant and the cutoff. Summarizing, we can say that it is hard to calculate properties of the ground state for realistic values of f and $k_{\max}$ with sufficient accuracy to assign confident limits to the result. But within the crude physical limits of validity the results of the model resemble experimental findings for the ground state.

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[1] This is not to be confused with the renormalized coupling constant $f_r^2/4\pi = \mathfrak{r}_2 f^2/4\pi \sim 0.1$.

CHAPTER 18

Pion Scattering

18.1. Introduction. In this chapter we turn our attention to the scattering of pions by nucleons. In the static limit, which we are using, the Hamiltonian can absorb and emit only angular-momentum P-wave mesons. Hence only the $L = 1$ ($J = \frac{1}{2}$ or $\frac{3}{2}$) phase shifts will differ from zero. Experimentally[1] it is known that there are also nonzero S-wave phases, as well as D waves at higher energies, but the dominant contribution in the energy region in which the static model makes sense arises from P waves. Part of the other phase shifts can be ascribed to recoil (kinematical) corrections, but this is not the complete story. Of the P waves, only the $J = \frac{3}{2}$, $T = \frac{3}{2}$ phase shift is important, since it has a resonance in a region where the static limit may still be sensible. This resonance is not unexpected on the basis of the classical and strong-coupling models we have discussed earlier. We shall see that it can be predicted quite naturally on the basis of the Low equations,[2] which we shall use to describe the scattering. This method has the advantage that the detailed form of the mesonic wave function is not needed. Although a complete expression for the S matrix is not obtained, the main features of pion scattering can be deduced. More important is the fact that the method to be described is the only one[3] which is not based on uncontrolled mathematical approximations.

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[1] See, e.g., H. A. Bethe and F. de Hoffmann, "Mesons and Fields," vol. II, chap. 3B, Row, Peterson & Company, Evanstan, Ill., 1955.

[2] F. E. Low, *Phys. Rev.*, **93**:1392 (1955).

[3] Relativistic dispersion relations can be shown to reduce to the equations to be discussed, in the limit of infinite nucleon mass and neglect of nucleon-antinucleon-pair creation.

18.2. The Scattering Matrix.[1] The derivation of the Low equation is analogous to that developed in Chap. 14 for the Lee model. Additional complications are introduced here by the presence of the 36 independent states of a nucleon and a P-wave pion, $|\,\text{in}, N + \pi\rangle$, at a fixed energy. These states can be written in our standard way as

$$|\,\text{in}, N + \pi_{\mathbf{k}}\rangle = A^{\dagger}_{j\alpha}(k)\,|\,N\rangle \equiv A^{\dagger}_{K}\,|\,N\rangle \tag{18.1}$$

For this development the expansion of the field in angular-momentum variables $A_{j\alpha}(k)$ is most appropriate. To avoid crowding of subscripts, we shall often make use of the notation introduced in (18.1); that is, we shall let the subscript K denote the nine possible angular-momentum and isospin indices of the meson, as well as the (semi-) continuous variable k. Introducing a single subscript ξ to distinguish the four possible nucleon states, we can write (18.1) as

$$|\,\text{in}, \mathbf{k}\xi\rangle = A^{\dagger}_{K}\,|\,\xi\rangle \tag{18.2}$$

The generalization of this description for a state with n real mesons, which will be needed shortly, is

$$|\,\text{in}, n, \xi\rangle \equiv |\,\text{in}, \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \xi\rangle = A^{\dagger}_{K_1}A^{\dagger}_{K_2}\cdots A^{\dagger}_{K_n}\,|\,\xi\rangle$$

The scattering matrix relates the out and in states. In the notation introduced above, this 36 × 36 matrix is

$$S_{\mathbf{k}'\xi',\mathbf{k}\xi} = \langle\text{out}, \mathbf{k}'\xi'\,|\,\text{in}, \mathbf{k}\xi\rangle = \langle\xi'\,|\,B_{K'}A^{\dagger}_{K}\,|\,\xi\rangle \tag{18.3}$$

To obtain the relation between the in and out meson operators, we use (15.10) rewritten in terms of angular-momentum operators:

$$a^{\dagger}_{i\alpha}(k,t) \equiv a^{\dagger}_{K}(t) = A^{\dagger}_{K}(t) + if\int_{-\infty}^{t} \frac{dt'\; k^2\rho(k)\sigma_i(t')\tau_\alpha(t')e^{i\omega(t-t')}}{(12\omega\pi^2)^{\frac{1}{2}}}$$

Similarly, in terms of the outgoing meson operator B, we have

$$a^{\dagger}_{K}(t) = B^{\dagger}_{K}(t) - if\int_{t}^{\infty} dt'\, \frac{k^2\rho(k)\sigma_i(t')\tau_\alpha(t')e^{i\omega(t-t')}}{(12\omega\pi^2)^{\frac{1}{2}}}$$

so that finally (at $t = 0$)

$$A^{\dagger}_{K}(0) = B^{\dagger}_{K}(0) - i\int_{-\infty}^{\infty} dt\; e^{-i\omega t}V_K(t) \tag{18.4}$$

where the source term has been abbreviated by

$$V_K(t) = f\,\frac{\rho(k)k^2}{(12\omega\pi^2)^{\frac{1}{2}}}\,[\sigma(t)\tau(t)]_K \tag{18.5}$$

The operator V is hermitian, $V_K(t) = V^{\dagger}_{K}(t)$.

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[1] To increase the legibility of the formulas to follow, we shall henceforth set the meson mass μ equal to unity. This means that all energies are measured in units of meson masses and all lengths in Compton wavelengths of the pion.

Substitution of (18.4) into (18.3) gives

$$\begin{aligned} S_{\mathbf{k}'\xi',\mathbf{k}\xi} &= \langle \xi' \mid B_{K'} B_K^\dagger \mid \xi \rangle - i \langle \text{out}, \mathbf{k}'\xi' \mid \int_{-\infty}^{\infty} V_K(t) e^{-i\omega t}\, dt \mid \xi \rangle \\ &= \delta_{\xi\xi'} \delta_{KK'} - 2\pi i \delta(\omega - \omega') \langle \text{out}, K'\xi' \mid V_K(0) \mid \xi \rangle \\ &= \delta_{\xi\xi'} \delta_{KK'} - 2\pi i \delta(\omega - \omega') T_{K'\xi',K\xi} \end{aligned} \tag{18.6}$$

To obtain the final form of (18.6), we made use of the explicit time dependence of $V_K(t)$ in the Heisenberg representation [see (2.18)] and integrated over time.

For inelastic processes, the S and T matrices differ only by the energy conservation δ function:

$$\begin{aligned} S_{n,K\xi} &= \langle \text{out}, \mathbf{k}_1', \mathbf{k}_2', \ldots, \mathbf{k}_n', \xi' \mid \text{in}, \mathbf{k}, \xi \rangle = -2\pi i \delta\Big(\sum_{i=1}^{n} \omega_i - \omega\Big) \\ &\quad \times \langle \text{out}, K_1', K_2', \ldots, K_n', \xi' \mid V_K(0) \mid \xi \rangle \equiv -2\pi i \delta(E_n - \omega) T_{n,K\xi} \end{aligned} \tag{18.7}$$

By means of the field equations, (18.6) can be rewritten on the energy shell ($\omega = \omega'$) as

$$\begin{aligned} T_{K'\xi',K\xi} &= \langle \text{out}, K'\xi' \mid V_K \mid \xi \rangle = \langle \xi' \mid \left(a_{K'} - i \int_0^{\infty} dt\, e^{i\omega t} V_{K'}(t) \right) V_K \mid \xi \rangle \\ &= \langle \xi' \mid V_K \left(B_{K'} + i \int_0^{\infty} V_{K'}(t) e^{i\omega t}\, dt \right) \mid \xi \rangle - i \langle \xi' \mid \int_0^{\infty} e^{i\omega t} V_{K'}(t)\, dt\, V_K \mid \xi \rangle \\ &= -i \int_0^{\infty} \langle \xi' \mid [V_{K'}(t), V_K(0)] \mid \xi \rangle e^{i\omega t}\, dt \end{aligned} \tag{18.8}$$

It is in the last form of this equation that some simplifications become apparent. We note that the momentum dependence of the T matrix can be factored out, leaving a matrix of spin and isospin operators between physical nucleons. This factorization is a special property of the static model, which we shall call on in subsequent sections.

To carry the discussion further, we rewrite the T matrix for elastic scattering by introducing a complete set of intermediate states $\mid \text{out}, n \rangle$, as we did in the Lee model. If we then use

$$\langle \xi' \mid V_{K'}(t) \mid \text{out}, n \rangle = e^{-iE_n t} T^*_{n,K'\xi'} \tag{18.9}$$

we obtain[1]

$$T_{K'\xi',K\xi} = -\mathop{\mathrm{S}}_{n} \left(\frac{T^*_{n,K'\xi'} T_{n,K\xi}}{E_n - \omega - i\epsilon} + \frac{T^*_{n,K\xi'} T_{n,K'\xi}}{E_n + \omega} \right) \tag{18.10}$$

Note that in the second term of this equation the meson indices are crossed but the nucleon ones are not.

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[1] The sum over n includes a sum over isospin and spin states as well as an energy integration in the continuum. That is,

$$\mathop{\mathrm{S}}_{n} = \sum_{J,T,J_z,T_z} \int dk$$

18.3. Properties of the Scattering Matrices. The form of the nonlinear equation (18.10) allows us to draw several important conclusions.

a. Unitarity. The condition $S^\dagger S = 1$, which always holds, can be explicitly demonstrated when one physical meson is present, as follows:

$$\begin{aligned}\langle \text{in}, K'\xi' \mid S^\dagger S \mid \text{in}, K\xi\rangle &= \mathop{\mathsf{S}}_{n} \langle \text{in}, K'\xi' \mid S^\dagger \mid n\rangle\langle n \mid S \mid \text{in}, K\xi\rangle \\ &= \delta_{K'K}\,\delta_{\xi'\xi} + 2\pi i\delta(\omega' - \omega)[T^*_{K\xi,K'\xi'} - T_{K'\xi',K\xi} \\ &\quad - 2\pi i \mathop{\mathsf{S}}_{n} T^*_{n,K'\xi'}T_{n,K\xi}\,\delta(E_n - \omega)] = \delta_{K'K}\,\delta_{\xi'\xi} \qquad (18.11)\end{aligned}$$

The vanishing of the bracket in (18.11) can be shown by substituting (18.10) into the difference $T^*_{K\xi,K'\xi'} - T_{K'\xi',K\xi}$. The second term of (18.10) cancels out, and the first term gives a sum

$$\mathop{\mathsf{S}}_{n}\left(\frac{T^*_{n,K'\xi'}T_{n,K\xi}}{E_n - \omega - i\epsilon} - \frac{T^*_{n,K'\xi'}T_{n,K\xi}}{E_n - \omega + i\epsilon}\right) = \mathop{\mathsf{S}}_{n} 2\pi i\delta(E_n - \omega)T^*_{n,K'\xi'}T_{n,K\xi} \qquad (18.12)$$

In a similar manner we can demonstrate that $SS^\dagger = 1$. The nonlinear character of the first term of (18.10) is thus connected with the unitarity of S. Although the second term does not contribute directly to the unitarity condition, it is vital to ensure the crossing symmetry discussed next.

b. Crossing Symmetry. The crossing symmetry of the T matrix is most conveniently expressed by considering separately the dependence of T on the variable ω in the denominator, in contradistinction to the dependence on $k^2\rho(k)\omega^{-\frac{1}{2}}$, which is explicitly known. To this end, we define t as

$$t_{\xi'K',\xi K}(z) = -\mathop{\mathsf{S}}_{n}\left(\frac{T^*_{n,\xi'K'}T_{n,\xi K}}{E_n - z} + \frac{T^*_{n,\xi'K}T_{n,\xi K'}}{E_n + z}\right) \qquad (18.13)$$

which depends on the complex variable z and has the property that

$$T_{\xi'K',\xi K} = \lim_{z\to\omega+i\epsilon} t_{\xi'K',\xi K}(z) \qquad (18.14)$$

In this physical region the $i\epsilon$ which appears [e.g., after substitution of (18.14)] in the denominator of the second term in (18.13) plays no role. It is needed, however, for the hermiticity condition

$$t^*_{K\xi,K'\xi'}(z^*) = t_{K'\xi',K\xi}(z) \qquad (18.15a)$$

and for the crossing symmetry of the T matrix, which is formulated mathematically as

$$t_{\xi'K',\xi K}(z) = t_{\xi'K,\xi K'}(-z) \qquad (18.15b)$$

This theorem of analytic continuation is directly verifiable by substitution into (18.13). It does not depend explicitly on the form of V_K and can be shown to be valid for any meson-nucleon coupling with both absorption and emission[1] (i.e., not for the Lee model but for the neutral scalar theory, etc.). The symmetry is related to the fact that the theory is invariant under interchange of ingoing and outgoing mesons in the sense expressed explicitly by (18.15*b*). It has no real intuitive basis, since the connection is between a physical scattering amplitude and one in an unphysical negative-energy region. The theorems (18.15) are rather concerned with the analytic properties of the t matrix for real or complex energies.

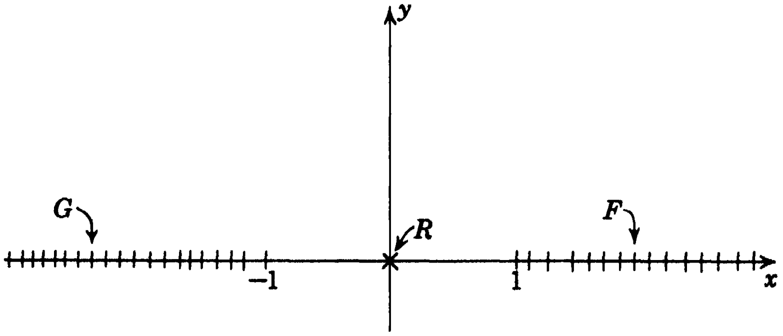

Fig. 18.1. Singularities of the matrix t in the complex z plane.

c. Poles, Branch Points, and Branch Cuts. The analytical properties of the t matrix are explicitly given by (18.13) and are determined by the energy spectrum of the intermediate states n. The sum over these states consists of one contribution at $E_n = 0$ from the ground state and a continuum from 1 to ∞. Thus t has the spectral form

$$t_{\xi' K',\xi K}(z) = \frac{R_{\xi' K',\xi K}}{z} + \int_1^{\infty} dE \left[\frac{F_{\xi' K',\xi K}(E)}{E - z} + \frac{G_{\xi' K',\xi K}(E)}{E + z} \right] \qquad (18.16)$$

where the first term arises from the ground state and F and G are weighting functions. These can be found from (18.13) and (18.15),

$$\begin{aligned} F_{\xi' K',\xi K}(E) &= \frac{1}{2\pi i} \lim_{z \to E + i\epsilon} [t_{\xi' K',\xi K}(z) - t_{\xi' K',\xi K}(z^*)] \\ &= G_{\xi' K,\xi K'}(E) \end{aligned} \qquad (18.17)$$

Hence t has a pole at the origin with residue R given by

$$R_{\xi' K',\xi K} = \mathop{\mathrm{S}}_{\zeta} (\langle \xi' \mid V_{K'} \mid \zeta \rangle \langle \zeta \mid V_K \mid \xi \rangle - \langle \xi' \mid V_K \mid \zeta \rangle \langle \zeta \mid V_{K'} \mid \xi \rangle) \qquad (18.18)$$

where the summation over ζ is to be taken over the four physical ground states. The matrix t also has cuts from 1 to ∞ and −1 to −∞, as indicated in Fig. 18.1. The contribution of the negative-axis cut arises from the crossing symmetry and was thus absent in the Lee model.

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[1] M. Gell-Mann and M. L. Goldberger, *Phys. Rev.*, **96**:1433 (1954).

18.4. Low- and High-energy Limits of Elastic Scattering. As in the Lee model, it is possible to define consistently a renormalized coupling constant f_r such that the physically not accessible limit of $t_{\xi'K',\xi K}$ as $z \to 0$ is given by the Born approximation, except that f_r replaces f. In the limit given above, the singular term R/z dominates all others. Remembering the definition of V_K given by (18.5), we see that only matrix elements of $\sigma\tau$ between physical nucleon states are involved. These have already been studied in the last section, and we find

$$R_{\xi'K',\xi K} = f^2 \mathfrak{r}_2^2 \frac{k^4\rho^2(k)}{12\omega\pi^2} (\xi' \mid (\sigma\tau)_{K'}(\sigma\tau)_K - (\sigma\tau)_K(\sigma\tau)_{K'} \mid \xi) \qquad (18.19)$$

where $\mathop{\mathrm{S}}_{\xi}$ is implied by the matrix multiplication of $(\sigma\tau)_{K'}$ and $(\sigma\tau)_K$. Aside from the factor $\mathfrak{r}_2^2$, this is the perturbation-theory result. The matrix element for the latter can be computed from the two (Feynman) diagrams of Fig. 18.2, and we find

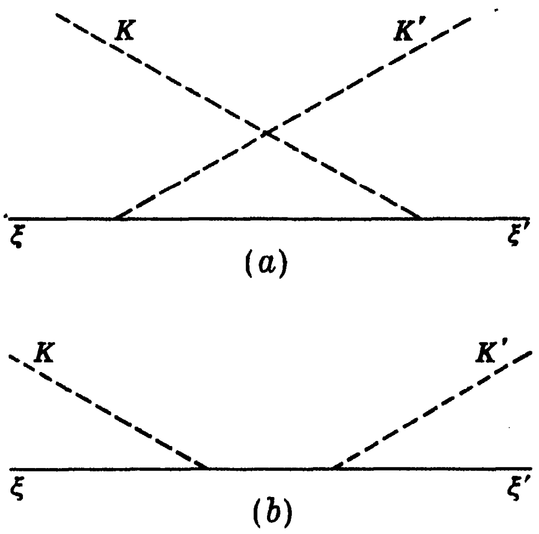

Fig. 18.2. Graphs for the lowest-order perturbation calculation of the matrix R.

$$\frac{R_{\xi'K',\xi K}}{\omega} = \frac{H'(K')H'(K) - H'(K)H'(K')}{\omega}$$

which corresponds to (18.19) with $\mathfrak{r}_2^2 = 1$. Hence

$$\begin{gathered} f_r = \mathfrak{r}_2 f \\ f\langle\xi' \mid \sigma\tau \mid \xi\rangle = f_r(\xi' \mid \sigma\tau \mid \xi) \end{gathered} \qquad (18.20)$$

Exactly as in the Lee model, the interpretation of

$$\lim_{z\to 0} t = \left(\frac{f_r}{f}\right)^2 \lim_{z\to 0} t_{\mathrm{Born}}$$

is that, in the idealized limit, the time between emission and absorption of the external meson takes much longer than that of all virtual pions. The intermediate nucleon is therefore a real one for all essential purposes, and we obtain the Born approximation, except for a reduction factor $\mathfrak{r}_2$. This factor has the same probability interpretation as in the Lee model. The constant f_r represents the interaction strength of the components of the physical nucleon weighted with the probability with which they occur.

In the high-energy limit, it is possible to neglect E_n relative to z in the denominator of (18.13). Thus

$$\begin{aligned}\lim_{z\to\infty} t_{\xi'K',\xi K}(z) &= \frac{1}{z} \mathop{\mathrm{S}}_{n} (\langle \xi' \mid V_{K'} \mid \text{out}, n\rangle\langle \text{out}, n \mid V_K \mid \xi\rangle \\ &\quad - \langle \xi' \mid V_K \mid \text{out}, n\rangle\langle \text{out}, n \mid V_{K'} \mid \xi\rangle) \\ &= \frac{1}{z} \langle \xi' \mid [V_{K'}, V_K] \mid \xi\rangle \end{aligned} \tag{18.21}$$

This is of the form of t in the Born approximation, except that the matrix element is taken with the physical nucleon states rather than with the bare states. That is to say, in the limit considered, the scattering amplitude is given by Born-approximation scattering from the various bare nucleon states multiplied by the amplitudes with which they occur in the physical nucleon. These can be easily found by observing that, aside from a known momentum dependence, the commutator in (18.21) is proportional to

$$[\tau_\alpha \sigma_j, \tau_{\alpha'} \sigma_{j'}] = 2i(\delta_{\alpha\alpha'} \epsilon_{jj'k} \sigma_k + \delta_{jj'} \epsilon_{\alpha\alpha'\gamma} \tau_\gamma) \tag{18.22}$$

where $\epsilon_{\alpha\beta\gamma}$ is ± 1 according to whether α, β, γ is an even $(+1)$ or an odd (-1) permutation of 1, 2, 3. We therefore find, with the aid of (17.3), that in the limit of infinite energy the t matrix is $\mathfrak{r}_1$ times the Born approximation:

$$\begin{aligned}\lim_{z\to\infty} z t_{\xi'K',\xi K}(z) &= \mathfrak{r}_1 f^2 \frac{k^4 \rho^2(k)}{12\omega\pi^2} (\xi' \mid [(\sigma\tau)_{K'}, (\sigma\tau)_K] \mid \xi) \\ &= \mathfrak{r}_1 \lim_{z\to\infty} z t_{\text{Born}} \end{aligned} \tag{18.23}$$

Whereas the zero-energy theorem is perhaps the most important instrument for linking theory and experiment, as we shall see, the high-energy limit is of purely academic interest, since it is outside the realm of validity of the model. Clearly in this limit, the neglect of recoil and pair creation (and the use of a finite source size) cannot make sense.

18.5. Diagonalization of the T Matrix. To use the formalism developed at the end of Chap. 8, we shall now diagonalize the 36 × 36 matrix t. Because the interaction conserves angular momentum and isospin (e.g., $J^{\text{out}} = J = J^{\text{in}}$, $T^{\text{out}} = T = T^{\text{in}}$), we expect that this can be done by transforming from the one-meson states $\mid \xi K\rangle$ to a representation in which T^2, T_z, J^2, J_z are diagonal.[1] Indeed, since these variables plus the energy determine the one-meson states completely, we have

$$\begin{aligned}\langle \text{out}, T', T'_z, J', J'_z \mid \text{in}, T, T_z, J, J_z\rangle &= \langle \text{in}, T', T'_z, J', J'_z \mid S \mid \text{in}, T, T_z, J, J_z\rangle \\ &= \delta_{T,T'}\, \delta_{T_z,T'_z}\, \delta_{J,J'}\, \delta_{J_z,J'_z} e^{i2\delta_{JT}}\, \delta(E - E') \frac{dE}{dk} \end{aligned} \tag{18.24}$$

---

[1] We hope that the reader will not confuse the matrix T and isospin T.

To relate the phase shift δ_{JT} to the T matrix, we refer to the end of Chap. 8. Since here $\mathfrak{S} = \int_0^\infty dk$, the factor $\pi g(E)$ in (8.30) is simply k/ω. Furthermore, we can project the T matrix into the corresponding eigenstates of J and T by means of projection operators $\mathfrak{P}^{JT}$,¶

$$T_{\xi'K',\xi K} = -\sum_{J,T} \mathfrak{P}^{JT}_{\xi'K',\xi K} \sin \delta_{JT} e^{i\delta_{JT}} \frac{k}{\pi\omega} \tag{18.25}$$

with

$$\mathfrak{P} = \mathfrak{P}^{\dagger} \tag{18.26}$$

and

$$\mathfrak{P}^{JT}_{\xi'K',\xi K} \equiv \langle \text{in}, \xi'K' \mid \mathfrak{P}^{JT} \mid \text{in}, \xi K \rangle \tag{18.27}$$

In the one-meson subspace, T and J assume only the values $\frac{1}{2}$ and $\frac{3}{2}$. Since δ_{JT} is δ_{TJ}, we have only three different phase shifts: $\delta_{\frac{1}{2},\frac{1}{2}}$, $\delta_{\frac{1}{2},\frac{3}{2}} = \delta_{\frac{3}{2},\frac{1}{2}}$, and $\delta_{\frac{3}{2},\frac{3}{2}}$. If we introduce the labels 1, 2, 3 for these phase shifts, then S assumes the form shown in Fig. 18.3. The projection operators into

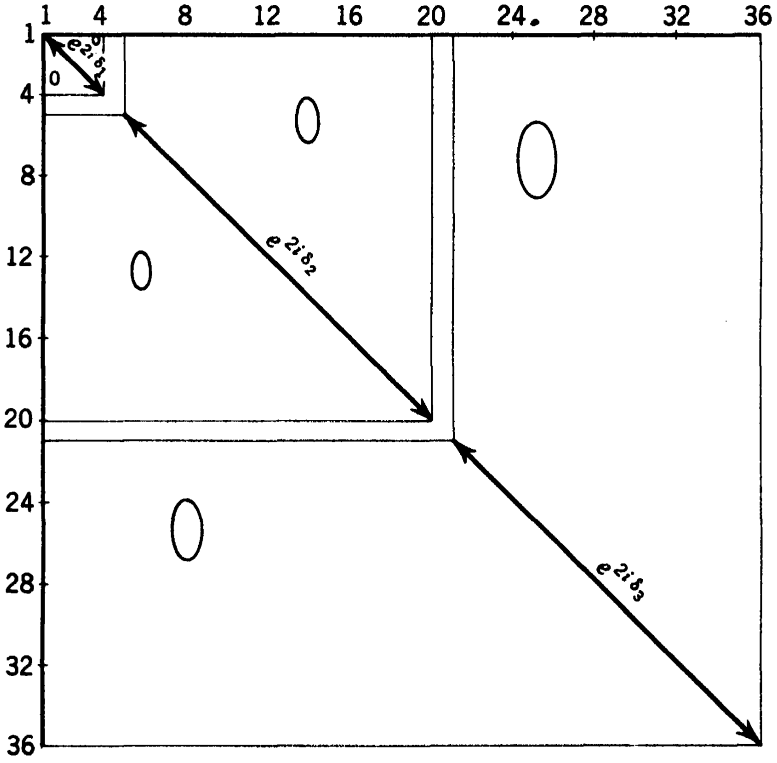

Fig. 18.3. Form of the S matrix. The off-diagonal elements are all zero, and the diagonal ones are as given in the figure.

¶ Compare (14.4a), which differs from (18.25) by anormalization fact or of $4\pi k^2$. The elastic-scattering cross section is

$$\sigma = \tfrac{1}{2} \times 8\pi^3\omega^2 k^{-4} \, |T|^2 = \sum_{JT} \mathfrak{P}^{JT} \frac{4\pi}{k^2} \, |\sin \delta_{JT} \, e^{i\delta_{JT}}|^2$$

these subspaces are constructed in the standard way with $\mathbf{T} = \mathbf{t} + \boldsymbol{\tau}/2$ and $\mathbf{J} = \boldsymbol{l} + \boldsymbol{\sigma}/2$:¶

$$\begin{aligned}
\mathfrak{P}^{(1)} &= \mathfrak{P}^{\frac{1}{2}\frac{1}{2}} = \tfrac{1}{9}(\tfrac{15}{4} - T^2)(\tfrac{15}{4} - J^2) = \tfrac{1}{9}(1 - \mathbf{t}\cdot\boldsymbol{\tau})(1 - \boldsymbol{\sigma}\cdot\boldsymbol{l}) \\
\mathfrak{P}^{(2)} &= \mathfrak{P}^{\frac{1}{2}\frac{3}{2}} + \mathfrak{P}^{\frac{3}{2}\frac{1}{2}} = \tfrac{1}{9}[(\tfrac{15}{4} - T^2)(J^2 - \tfrac{3}{4}) + (T^2 - \tfrac{3}{4})(\tfrac{15}{4} - J^2)] \\
&= \tfrac{1}{9}[(1 - \boldsymbol{\tau}\cdot\mathbf{t})(2 + \boldsymbol{\sigma}\cdot\boldsymbol{l}) + (2 + \boldsymbol{\tau}\cdot\mathbf{t})(1 - \boldsymbol{\sigma}\cdot\boldsymbol{l})] \qquad (18.28) \\
\mathfrak{P}^{(3)} &= \mathfrak{P}^{\frac{3}{2}\frac{3}{2}} = \tfrac{1}{9}(T^2 - \tfrac{3}{4})(J^2 - \tfrac{3}{4}) = \tfrac{1}{9}(2 + \boldsymbol{\tau}\cdot\mathbf{t})(2 + \boldsymbol{\sigma}\cdot\boldsymbol{l})
\end{aligned}$$

The normalization factors chosen are such that for the one-meson subspace

$$\mathfrak{P}^{(u)}\mathfrak{P}^{(v)} = \delta_{uv}\mathfrak{P}^{(v)} \qquad \text{where } u, v = 1, 2, 3 \qquad (18.29)$$

The matrix elements of $\mathbf{t}$ and $\boldsymbol{l}$ in the one-meson subspace simply correspond to the $l = 1$ representation and are [compare (15.12), (15.13) and (5.13)][1]

$$\begin{aligned}
\langle \text{in}, \xi, \alpha j \mid t_\beta^{(\text{in})} \mid \text{in}, j'\alpha', \xi' \rangle &= i\epsilon_{\beta\alpha\alpha'}\, \delta_{\xi\xi'}\, \delta_{jj'} \\
\langle \text{in}, \xi, \alpha j \mid l_u^{(\text{in})} \mid \text{in}, j'\alpha', \xi' \rangle &= i\epsilon_{ujj'},\, \delta_{\xi\xi'}\, \delta_{\alpha\alpha'}
\end{aligned} \qquad (18.30)$$

so that we have, e.g.,

$$\langle \text{in}, \xi, \alpha j \mid \boldsymbol{\sigma}^{(\text{in})} \cdot \boldsymbol{l}^{(\text{in})} \mid \text{in}, \alpha' j', \xi' \rangle = i\delta_{\alpha\alpha'}(\xi \mid \sigma_j\sigma_{j'} - \sigma_{j'}\sigma_j \mid \xi') \qquad (18.31)$$

Returning to our shorthand labeling, we can write for the matrix elements of the projection operators (18.28) in an angular-momentum representation

$$\begin{aligned}
\mathfrak{P}^{(1)}_{\xi'K',\xi K} &= \tfrac{1}{9}(\xi' \mid (\sigma\tau)_{K'}(\sigma\tau)_K \mid \xi) \\
\mathfrak{P}^{(2)}_{\xi'K',\xi K} &= \tfrac{1}{9}(\xi' \mid 3(\tau_{K'}\tau_K + \sigma_{K'}\sigma_K) - 2(\sigma\tau)_{K'}(\sigma\tau)_K \mid \xi) \qquad (18.32) \\
\mathfrak{P}^{(3)}_{\xi'K',\xi K} &= \tfrac{1}{9}(\xi' \mid 9\delta_{K'K} - 3\tau_{K'}\tau_K - 3\sigma_{K'}\sigma_K + (\sigma\tau)_{K'}(\sigma\tau)_K \mid \xi)
\end{aligned}$$

We shall shortly encounter expressions $\mathfrak{P}^{(u)}_{\xi'K,\xi K'}$ where the meson subscripts are exchanged but the nucleon subscripts ξ and ξ' remain in their usual order. Since this operation just changes the sign of $\boldsymbol{l}$ and $\mathbf{t}$, these operators can be expressed as linear combinations of the old ones:

$$\mathfrak{P}^{(u)}_{\xi'K,\xi K'} = \sum_v A^{(u)(v)}\, \mathfrak{P}^{(v)}_{\xi'K',\xi K} \qquad (18.32a)$$

We find by direct substitution that A is

$$A = \tfrac{1}{9}\begin{pmatrix} 1 & -2 & 4 \\ -8 & 7 & 4 \\ 16 & 4 & 1 \end{pmatrix} \qquad (18.33)$$

¶ Whereas $\mathbf{T}$ is constant in time, both $\mathbf{t}$ and $\boldsymbol{\tau}$ are time-dependent. It is their value at time $t = -\infty$ that is needed here.

[1] The labels j and α refer to the angular momentum and isospin of the meson, respectively.

The properties of A, namely, $A^2 = 1$, $\operatorname{Det} A \equiv |A| = -1$, $\operatorname{Tr} A = +1$, show that A has the eigenvalues $+1$, $+1$, -1.

With this development we can immediately obtain the zero-energy behavior of the T matrix. Thus, substitution of (18.29) and (18.32) into (18.19) gives

$$\begin{aligned} R_{\xi'K',K\xi} &= \frac{9f_r^2k^4\rho^2(k)}{12\omega\pi^2}\left(\mathfrak{P}^{(1)}_{\xi'K',\xi K} - A^{(1)(v)}\,\mathfrak{P}^{(v)}_{\xi'K',\xi K}\right) \quad . \\ &= -\frac{f_r^2k^4\rho^2(k)}{12\omega\pi^2}\,\mathfrak{P}^{(v)}_{\xi'K',\xi K}\lambda^{(v)} \end{aligned} \tag{18.34}$$

The quantity $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ is an eigenvector of A with eigenvalue -1:

$$\boldsymbol{\lambda} A = -\boldsymbol{\lambda}$$

$$\boldsymbol{\lambda} = (-8,-2,+4)$$

To obtain quantities with simple analytic properties, we introduce

$$t(\omega) = -\sum_v \mathfrak{P}^{(v)}h^{(v)}(\omega)\,\frac{\rho^2(k)k^4}{12\pi^2\omega} \tag{18.35}$$

for which we obtain, from (18.16) and (18.17),

$$\begin{aligned} \sum_v \mathfrak{P}^{(v)}h^{(v)} = \sum_v \Bigg\{ \mathfrak{P}^{(v)}\lambda^{(v)}\frac{f_r^2}{\omega} + \frac{1}{\pi}\int_1^\infty d\omega'\Bigg[&\frac{\mathfrak{P}^{(v)}\operatorname{Im} h^{(v)}(\omega')}{\omega'-\omega-i\epsilon} \\ &+ \sum_u \frac{\mathfrak{P}^{(v)}A^{(u)(v)}\operatorname{Im} h^{(u)}}{\omega'+\omega}\Bigg]\Bigg\} \end{aligned}$$

$$\text{or}\quad h^{(v)}(z) = \frac{f_r^2\lambda^{(v)}}{z} + \frac{1}{\pi}\int_1^\infty d\omega\left[\frac{\operatorname{Im} h^{(v)}(\omega)}{\omega - z} + \sum_u A^{(u)(v)}\,\frac{\operatorname{Im} h^{(u)}(\omega)}{\omega + z}\right] \tag{18.36}$$

Hence the functions $h^{(v)}(z)$ can be continued into the complex plane, where they have the properties

$$h^{(v)}(z) = h^{(v)*}(z^*) = \sum_u A^{(u)(v)}h^{(u)}(-z) \tag{18.37}$$

They are related to the phase shifts by

$$\lim_{z\to\omega+i\epsilon} h^{(v)}(z) = e^{i\delta_v(\omega)}\sin\delta_v(\omega)\,\frac{12\pi}{k^3\rho^2(k)} \tag{18.38}$$

as can be seen by comparing (18.35) with (18.25). The function h is related to the total cross section by

$$\operatorname{Im} h^{(v)} = \sigma_v\,\frac{1}{k\rho^2(k)} \tag{18.39}$$

if we define the total cross section for a P-wave channel by

$$\sigma_v = \frac{12\pi}{k^2} \sin^2 \delta_v$$

This definition is such that the cross section (8.34) for an unpolarized target and definite isospin is

$$\sigma_{\text{unpol}} = \tfrac{1}{2} \sum_{\xi=-\frac{1}{2}}^{+\frac{1}{2}} \operatorname{Im} T_{\xi'K',\xi K} = \tfrac{1}{3}\sigma^{(J=\frac{1}{2})} + \tfrac{2}{3}\sigma^{(J=\frac{3}{2})}$$

It should be noted that, for $\omega > 2$, particles can be produced and that the phase shifts describing the one-particle channel become complex. Nevertheless, (18.39) can still be used, since it depends only on the general equation (8.28). With these expressions we can cast the Low equation into the final form

$$\begin{aligned} h^{(1)}(\omega) &= -\frac{8f_r^2}{\omega} + \frac{1}{\pi}\int_1^\infty \frac{d\omega'}{k'\rho^2(k')}\left[\frac{\sigma_1(\omega')}{\omega' - \omega - i\epsilon} + \sum_v \frac{A^{(v)(1)}\sigma_v(\omega')}{\omega' + \omega}\right] \\ h^{(2)}(\omega) &= -\frac{2f_r^2}{\omega} + \frac{1}{\pi}\int_1^\infty \frac{d\omega'}{k'\rho^2(k')}\left[\frac{\sigma_2(\omega')}{\omega' - \omega - i\epsilon} + \sum_v \frac{A^{(v)(2)}\sigma_v(\omega')}{\omega' + \omega}\right] \qquad (18.40) \\ h^{(3)}(\omega) &= +\frac{4f_r^2}{\omega} + \frac{1}{\pi}\int_1^\infty \frac{d\omega'}{k'\rho^2(k')}\left[\frac{\sigma_3(\omega')}{\omega' - \omega - i\epsilon} + \sum_v \frac{A^{(v)(3)}\sigma_v(\omega')}{\omega' + \omega}\right] \end{aligned}$$

18.6. Relation of Low Equations to Experiment. In the Low equations (18.40), all quantities except the coupling constant f_r^2 are directly observable—at least in principle. The first terms on the right-hand side are the Born-approximation ones with the renormalized coupling constants and can be found only by extrapolations to the unphysical point $\omega = 0$. These terms are negative in the 1- and 2-states and positive in the 3-state, corresponding to a repulsion in the former and an attraction in the latter. This feature can be understood in terms of the lowest-order Feynman graphs, Fig. 18.2. Since Fig. 18.2*b* has an intermediate nucleon with $T = \frac{1}{2}$, $J = \frac{1}{2}$, it only contributes to scattering in the 1-state. Therefore, $\sigma^{(3)}$ is due solely to Fig. 18.2*a*, and the reason for the attraction is the same as for $n\pi^-$ scattering in the Lee model. In terms of perturbation theory, it arises because the intermediate state has higher energy than the initial one. On the other hand, the scattering in the 1-state arises mainly because of Fig. 18.2*b*, and in the 2-state the opposite sign of the p- and n-coupling constants to the π^0 changes the sign of the contribution from Fig. 18.2*a*.

In the total scattering amplitudes the integrals in (18.40) give an increase over the Born-approximation term in the 3-state and a decrease in the other states, because σ_3 is by far the largest of the cross sections. This is the same as in the scattering by a short-range potential, where the Born-approximation amplitude overestimates the effect of a repulsive potential and underestimates that of an attractive one.[1] According to the Born approximation, $h^{(1)} = -2h^{(3)}$, whereas empirically $|h^{(1)}| \ll |h^{(3)}|$, so that at physical energies the correction terms must be substantial. We shall calculate this effect, as a first approximation, by expanding the real parts of the integral in (18.40) in powers of ω. Keeping only the lowest term in the expansion, we obtain an "effective-range" approximation[2] to the phase shifts. Thus,

$$\mathrm{Re}\, h^{(u)} = \frac{f_r^2 \lambda^{(u)}}{\omega}(1 + \omega r_u) \tag{18.41a}$$

$$r_u = \frac{1}{\pi \lambda^{(u)} f_r^2} \int_1^\infty \frac{d\omega'}{\omega' k' \rho^2(k')} [\sigma_u(\omega') + A^{(v)(u)} \sigma_v(\omega')]$$

and

$$\frac{1}{\omega}\, \mathrm{Re}\, \frac{1}{h^{(u)}} = \cot \delta_u \frac{k^3 \rho^2(k)}{12\pi\omega} \approx \frac{1 - \omega r_u}{f_r^2 \lambda^{(u)}} \tag{18.41b}$$

A plot of $k^3\rho^2(k) \cot (\delta_u/\lambda^{(u)}3\omega)$ as a function of ω should approach a straight line at low energies. The intercept of such a "Chew-Low" plot extrapolated to $\omega = 0$ is $4\pi/f_r^2$ and should be the same for all phase shifts. Experimentally, the 1- and 2-phase are unfortunately too small to be measured to sufficient accuracy. In Fig. 18.4 we plot the middle part of (18.41*b*), with $\rho(k) = 1$ and the experimental 3-phase,[3] as a function of ω. The experimentally determined points are seen to lie on a reasonably straight line which intercepts the ordinate at

$$\frac{f_r^2}{4\pi} = 0.087 \pm 0.01 \tag{18.42}$$

---

[1] With this analogy in mind, an estimate of the departure from the zero-energy form (renormalized Born approximation) has been given by V. F. Weisskopf, *Phys. Rev.*, **116**:1615 (1959), under the assumption that the logarithmic derivative of the wave function $\langle 1 \mid \phi \mid 0 \rangle$ at the source radius depends weakly on the energy. According to this procedure the 3-phase shift goes through 90° at about the right energy, whereas the other phase shifts remain small.

[2] See, e.g., J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," p. 62, John Wiley & Sons, Inc., New York, 1952. Since the Im $h^{(u)} \propto k^3$ it can be neglected at low energies.

[3] See S. W. Barnes, B. Rose, G. Giacomelli, J. King, K. Miyake, and K. Kinsey, *Phys. Rev.*, **117**:226 (1960), from which Fig. 18.4 has been taken. The experimental points arise from many sources, which are listed in the above reference. See also S. J. Lindenbaum, *Ann. Rev. Nuclear Sci.*, **7**:317 (1957).

According to (18.40) the slope of the straight line is the meson mass times the effective range r_3 and can be expressed in terms of integrals over the total cross section. This actually holds within the expected accuracy, as will be shown in the next section. Furthermore, the phase shift passes through 90° at $\omega = \omega_r = 1/r_3 \approx 2.1$. That this also agrees with experiment can be seen by a comparison with Fig. 15.1, in which the total cross section for $\pi^+ + p$ scattering is seen to pass through a resonance at a laboratory kinetic energy of ~190 Mev, corresponding to a center of mass $\omega \approx 2.1$.

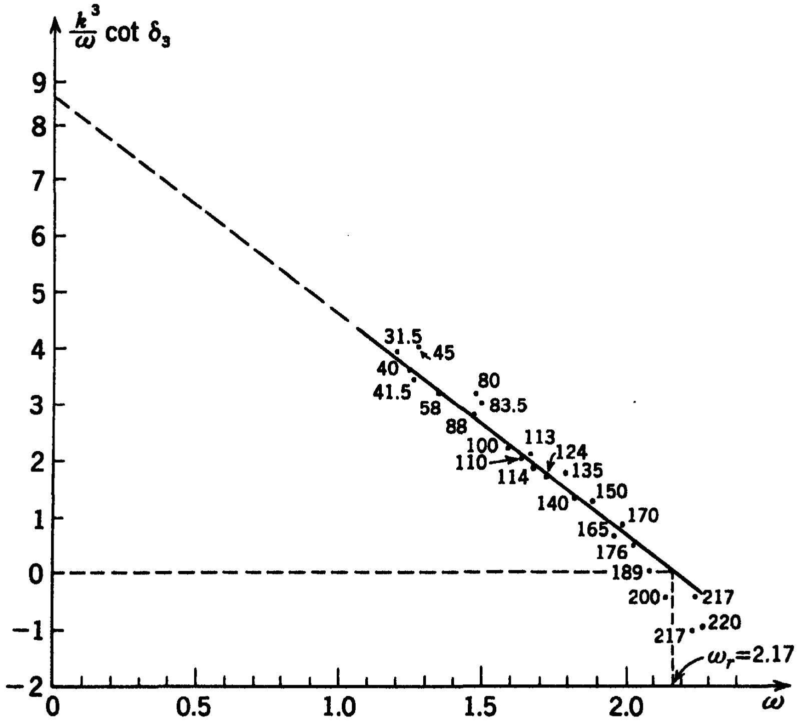

Fig. 18.4. Chew-Low plot of (18.41*b*). The intercept of the straight line with the ordinate is $3/[4(f_r^2/4\pi)]$ and with the abscissa is r_3. The curve is taken from S. W. Barnes, B. Rose, G. Giacomelli, J. King, K. Miyake, and K. Kinsey, *Phys. Rev.*, **117**:226 (1960). The sources for the experimental points that appear in the figure are listed in the above reference. The numbers given are the experimental energies.

18.7. Approximate Solution of the Low Equation. Having determined f_r^2 from experiment, we should like to find $\delta_u(\omega)$ without putting in further experimental data. This can be done, but only with occasional appeal to known empirical facts. First of all, we know that within the region of validity of our model, the scattering is mainly elastic. This is rigorously so for $\omega < 2$, but we shall assume it to hold for all energies. Then the phase shifts δ_u are real, and we can write

$$\sigma_u = |h_u|^2 \frac{\rho^4(k)k^4}{12\pi} \tag{18.43}$$

This approximation is tantamount to including only zero- and one-meson states in the sum over the intermediate states.[1]

As was done in the corresponding problem of the Lee model, it is useful to introduce the inverse of h:

$$g_u = \frac{f_r^2 \lambda^{(u)}}{z h^{(u)}(z)} \tag{18.44}$$

Since $h^{(u)}(z)$ has no zeros for complex z (there $\operatorname{Im} h > 0$) and $zh(z)$ is finite at $z = 0$, g is analytic save for cuts along the real axis from 1 to ∞ and -1 to $-\infty$.[2] The factors in (18.44) have been chosen so that $g_u(0) = 1$ and hence, in analogy with (18.16), g can be written[3]

$$g_u(z) = 1 - \frac{z}{\pi} \int_1^\infty d\omega \left[\frac{\mathfrak{F}_u(\omega)}{\omega - z} + \frac{\mathfrak{G}_u(\omega)}{\omega + z} \right] \tag{18.45a}$$

The real weighting functions $\mathfrak{F}_u$ and $\mathfrak{G}_u$ correspond to the imaginary parts of g on the positive and negative real axes. We readily obtain from (18.45*a*), if we make use of (18.37), e.g., $g_u(z^*) = g_u^*(z)$,

$$\begin{aligned} g_u(\omega + i\epsilon) - g_u(\omega - i\epsilon) &= 2i \operatorname{Im} g_u(\omega) = -2i\omega\mathfrak{F}_u(\omega) \\ g_u(-\omega - i\epsilon) - g_u(-\omega + i\epsilon) &= 2i \operatorname{Im} g_u(-\omega) = 2i\omega\mathfrak{G}_u(\omega) \end{aligned} \tag{18.45b}$$

Within the one-meson approximation, we furthermore find, from (18.38),

$$\operatorname{Im} g_u(\omega) = -\frac{f_r^2}{4\pi} \frac{k^3 \rho^2(k)}{\omega} \frac{\lambda^{(u)}}{3} \tag{18.46}$$

and thus obtain $\mathfrak{F}_u(\omega)$ without knowing $\delta_u(\omega)$. To determine $\mathfrak{G}_u$, on the other hand, we need the phase shifts in an unphysical region. This

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[1] This procedure should not be confused with the Tamm-Dancoff approximation, where the expansion of the physical nucleon is in terms of the number of mesons around a *bare* nucleon, not a physical one. Here, the expansion of a physical nucleon and a meson is made in terms of *physical* states. In this way we obtain the correct low-energy behavior.

[2] There may be poles on the real axis between -1 and 1. Their significance is discussed by Castillejo, Dalitz, and Dyson. (See the first reference on page 144.) We shall not examine these singularities.

[3] In general, the power of z in front of the integral need not be 1. However, if

$$\int_1^\infty \frac{d\omega}{k\rho^2(k)} \sigma(\omega) < \infty$$

we see that $\lim_{z \to \infty} g_u = \text{constant}$. Hence the choice z^1, which also leads to the correct perturbation-theory result.

can be obtained from the crossing symmetry (18.37), which becomes, for g_u,

$$\frac{1}{g_u(-z)} = \frac{zh^{(u)}(-z)}{f_r^2\lambda^{(u)}} = -\frac{1}{\lambda^{(u)}}A^{(u)(v)}\lambda^{(v)}\frac{1}{g_v(z)} \equiv B_{uv}\frac{1}{g_v(z)}$$

$$B_{uv} = -\frac{\lambda^{(v)}}{\lambda^{(u)}}A^{(u)(v)} = \frac{1}{9}\begin{pmatrix} -1 & 2 & 8 \\ 8 & -7 & 8 \\ 8 & 2 & -1 \end{pmatrix} \tag{18.47}$$

As for the matrix $A^{(u)(v)}$, the eigenvalues of B_{uv} are ± 1 and $B^2 = 1$.

This equation gives only the $\operatorname{Im} g^{-1}(-z)$; to find $\operatorname{Im} g(-z)$, we also need the $\operatorname{Re} g^{-1}(-z)$, which is not obtainable directly. Thus, even in the one-meson approximation, it is not possible to find an exact solution. However, an approximate solution has been suggested by Chew and Low,[1] who replace B by B',

$$B \approx B' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = B - \frac{1}{9}\begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{pmatrix} \tag{18.48}$$

which differs from B by a small numerical displacement and also satisfies

$$B'^2 = 1 \qquad \operatorname{Tr} B' = 1 \qquad |B'| = 1 \tag{18.49}$$

With this approximation, the problem is soluble for g_1 and g_3, but not yet for g_2. From (18.47) we find

$$\begin{aligned} g_1(\mp z) &= g_3(\pm z) \\ g_2^{-1}(-z) &= g_1^{-1}(z) - g_2^{-1}(z) + g_3^{-1}(z) \end{aligned} \tag{18.50}$$

and therefore (18.45*b*) predicts

$$\begin{aligned} \mathfrak{G}_1(\omega) &= -\mathfrak{F}_3(\omega) = -\frac{\lambda_3 k^3\rho^2(k)f_r^2}{12\pi\omega^2} \\ \mathfrak{G}_3(\omega) &= -\mathfrak{F}_1(\omega) = -\frac{\lambda_1 k^3\rho^2(k)f_r^2}{12\pi\omega^2} \end{aligned} \tag{18.51}$$

There are no real mathematical arguments to justify (18.48). However, using empirical phase shifts, we can show that what has been omitted is at least not large compared with what was taken into account. Hence, within our rough model, the procedure may be a reasonable illustration, but the following numbers should not be taken too seriously. We find

---

[1] G. F. Chew, Theory of Pion Scattering and Photoproduction, in "Handbuch der Physik," Springer-Verlag, Berlin (to be published).

$$
\begin{aligned}
g_1(\omega) &= 1 + \frac{f_r^2}{4\pi}\frac{\omega}{3\pi}\int_1^\infty d\omega'\,\frac{k'^3\rho^2(k')}{\omega'^2}\left(\frac{8}{\omega'-\omega-i\epsilon}+\frac{4}{\omega'+\omega}\right)\\
g_3(\omega) &= 1 - \frac{f_r^2}{4\pi}\frac{\omega}{3\pi}\int_1^\infty d\omega'\,\frac{k'^3\rho^2(k')}{\omega'^2}\left(\frac{4}{\omega'-\omega-i\epsilon}+\frac{8}{\omega'+\omega}\right)
\end{aligned}
\tag{18.52a}
$$

and therefore an expansion in powers of ω gives (at low energies)

$$
\begin{aligned}
h^{(1)}(\omega) &= -\frac{8}{\omega}f_r^2\frac{1}{1+\dfrac{f_r^2}{4\pi}\dfrac{4\omega}{\pi}\displaystyle\int_1^\infty d\omega'\,\frac{k'^3\rho^2(k')}{\omega'^3}}\\
h^{(3)}(\omega) &= \frac{4f_r^2}{\omega}\frac{1}{1-\dfrac{f_r^2}{4\pi}\dfrac{4\omega}{\pi}\displaystyle\int_1^\infty d\omega'\,\frac{k'^3\rho^2(k')}{\omega'^3}}
\end{aligned}
\tag{18.52b}
$$

The form of the scattering amplitudes for $h^{(1)}$ is familiar to us[1] from pair theory with repulsion or $p\pi^-$ scattering in the Lee model. For $h^{(3)}$ it is like that of pair theory with attraction or $n\pi^-$ scattering in the Lee model. It predicts a resonance in the $\frac{3}{2},\frac{3}{2}$-state for sufficiently strong coupling, as observed. To get an idea of the resonance energy, we calculate the effective ranges r_1 and r_3 predicted by (18.52). The range r_2 can then be found from $r_2 = -4(r_1 + r_3)$, which follows from the crossing symmetry (18.37) applied to (18.41*a*). In this way we obtain

$$
r_u = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \frac{4(f_r^2/4\pi)}{\pi}\int_1^\infty d\omega\,\frac{k^3\rho^2(k)}{\omega^3}
\tag{18.53}
$$

A somewhat different approximation from that considered above, of replacing B by B', consists in keeping only σ_3 in the Low equations (18.40). We readily find that in this case the value of r_1 is approximately the same as that given above but that $r_2 = r_1$ and $r_3 = -5r_1/4$ instead of $-r_1$; the main uncertainty is seen to arise in the range r_2. In either approximation the magnitude and sign of r_1 and r_3 turn out to be approximately the same. If we assume that the effective-range approximation is still approximately valid at the resonance energy ω_r, then we find $\omega_r = 1/r_3$. For a square cutoff, e.g.,

$$
\rho(k) = \begin{cases} 1 & \text{for } k < k_{\max} \\ 0 & \text{for } k > k_{\max} \end{cases}
$$

---

[1] Except for obvious changes due to our dealing with *P*-wave mesons.

we find

$$\omega_r \approx \left(\frac{f_r^2}{4\pi}\frac{4}{\pi}\,\omega_{\max}\right)^{-1} \tag{18.54}$$

The experimental value of $\omega_r \approx 2$ requires $\omega_{\max} \approx 5$, if we make use of (18.42). It is interesting to note that ω_r is exactly at the energy of the $\frac{3}{2},\frac{3}{2}$-level in the strong-coupling limit, which is given by (17.64):

$$E_3 = \frac{9}{\dfrac{f^2}{4\pi}\dfrac{4}{\pi}\,\omega_{\max}} \tag{18.55}$$

Since in this approximation (17.59) tells us that $f_r = f/3$, we find $E_3 = \omega_r$. The width Γ of the level is [see (12.9*a*)]

$$\begin{aligned}\Gamma &\equiv 2(\omega_r - \omega)\tan\delta = (\omega_r - \omega)\frac{k_r^3}{\omega_r}\frac{f_r^2}{4\pi}\frac{8}{3}(1 - r_3\omega)^{-1}\rho^2(k)\\ &= \frac{8}{3}\frac{f_r^2}{4\pi}k_r^3\rho^2(k) \sim 1.1\end{aligned} \tag{18.56}$$

which is cutoff-independent if k_r is not close to $k_{\max}$ and which is of the same form as that of the classical calculation (16.32). The shape of the cross section close to the resonance energy is of the usual type, which was discussed in Chap. 15.

18.8. Summary. To conclude this chapter, we shall point out the most important features of pion-nucleon scattering. At low energies, where $\sin\delta \propto f_r^2 k^3/\omega$, the cross section is expected to be proportional to

$$\sigma \propto f_r^4 \frac{k^4}{\omega^2} \tag{18.57}$$

and this is borne out experimentally, as shown in Fig. 15.1. In this region the "effective coupling strength" $(f_r k)$ is weak.

The form of the cross section can be understood as follows. Consider a box of volume L^3 which contains a meson of momentum k in addition to a nucleon at rest. Since the meson is not localized, its probability of getting within a sphere or radius ω^{-1} centered at the nucleon is $\sim\omega^{-3}L^{-3}$. Now, it turns out that the nucleon can emit and absorb mesons when they are within a distance ω^{-1}, which is the size of the cloud rather than of the source, and since it cannot distinguish the incoming meson from one of its own, it may absorb the former instead of the latter. The nucleon, however, is only meson-active for a fraction of time Δ, in the sense that there is only a certain probability that the nucleon emits and absorbs mesons.

If we represent the history of the nucleon graphically, as in Fig. 18.5, then the lengths of the meson lines are t $\approx \omega^{-1}$; these lines are distributed at random, and Δ is the ratio of their lengths to that of the nucleon line. In the scalar theories, we found that if there is predominantly one meson during the active period, then $\Delta \approx$ mean number of mesons $\approx g^2/4\pi$. For P-wave mesons, the interaction probability is strongly energy-dependent, and as long as $\Delta < 1$, it is equal to the effective coupling strength to a physical nucleon $\approx (f_r^2/4\pi)k^2$.

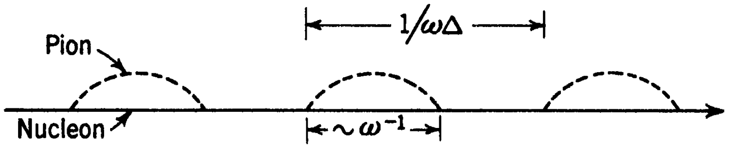

Fig. 18.5. Graph to represent a physical nucleon.

Now the absorption of the incoming meson violates energy conservation and can therefore last only for a time t $\sim 1/\Delta E = \omega^{-1}$. Within this time, the nucleon must[1] recreate a meson with energy ω, but not necessarily in the same direction. The event thus constitutes a scattering of the pion by the nucleon. Because the absorption and emission occur only within the quantum-mechanical fluctuation allowed by the uncertainty relation, we cannot tell when they happen within the interval t, and the events may occur in opposite order. The amplitudes for the two types of processes (emission-absorption, absorption-emission) interfere, since the intermediate state is not controlled by measurements, but the interference terms do not change the order of magnitude of the cross section (except in the 3-state at higher energies, near the resonance). For our qualitative argument, the various events can be considered to be independent of one another, and the probabilities for the separate stages multiplied (although this is not quite true quantum-mechanically). Then the probability for the total process is (η = probability)

$\eta_{\text{process}} = \eta$ (meson to get within meson cloud of nucleon) $\times\ \eta$ (that this occurs while the nucleon is meson-active) $\times\ \eta$ [for nucleon to emit (or absorb) a meson within time ω^{-1}]

$$\eta_{\text{process}} = \left(\frac{1}{\omega^3 L^3}\right)\left(\frac{f_r^2}{4\pi}k^2\right)\left(\frac{f_r^2}{4\pi}k^2\right)$$

The cross section is equal to this over-all probability divided by the

[1] This classical description is somewhat acausal, inasmuch as the nucleon absorbs the meson only when it knows that it will emit it in time.

flux of incoming mesons and multiplied by the rate at which mesons leave the cloud. The latter is equal to the velocity of the meson, k/ω, divided by the diameter of the cloud, ω^{-1}. Since the flux of ingoing mesons (e.g., the probability that they traverse unit area per unit time) is $\frac{k}{\omega}\frac{1}{L^3}$, the cross section is

$$\sigma \approx \left(\frac{f_r^2}{4\pi}k^2\right)^2 \frac{1}{\omega^3 L^3} k \frac{L^3\omega}{k} = \left(\frac{f_r^2}{4\pi}\frac{k^2}{\omega}\right)^2$$

Of course, the cross section is limited by the geometrical area of the cloud, $1/\omega^2$. When this limit is exceeded, our considerations become meaningless, since the various probabilities then exceed unity. The exact formal development shows that this limit will be attained only in the 3-state, and at higher energies. To the extent that the 3-resonance dominates the scattering, it is possible to make other simple statements concerning the cross section. The significance of the energy dependence near the resonance in terms of probabilities was discussed in Chap. 12. The angular distribution of the mesons can be inferred from our considerations following (16.37). If we take the axis of quantization along the direction of the incident meson momentum, then $J_z = s_z = \pm\frac{1}{2}$, and we found there that the angular distribution of the mesons in the $J = \frac{3}{2}$, $J_z = \pm\frac{1}{2}$ state is $\propto 1 + 3\cos^2\vartheta$. This should be compared with the classical result (16.33), which is also peaked forward and backward but which is much flatter. The predictions of charge independence are found by determining what part of the incoming and scattered states (e.g., $\pi^- p$) have $J = T = \frac{3}{2}$. For scattering on protons, we thus obtain, from (16.35),

$$\sigma_{\pi^+p\to\pi^+p} : \sigma_{\pi^-p\to\pi^0 n} : \sigma_{\pi^-p\to\pi^-p} = 9:2:1 \tag{18.58}$$

The experimental curves of Figs. 15.1 and 18.6 show that the above predictions are fairly well satisfied. The maximum of the curve for π^+p scattering in Fig. 15.1 demonstrates that the resonance is indeed in a $J = \frac{3}{2}$ state, since the cross section reaches its maximum allowable value of $8\pi/k^2$.

The simple static model we have expounded also predicts that the phases $\delta_{\frac{1}{2},\frac{3}{2}}$ and $\delta_{\frac{3}{2},\frac{1}{2}} = \delta_2$ will be equal, negative, and small:

$$\tan\delta_2 \approx -\frac{f_r^2}{4\pi}\frac{2}{3}\frac{k^3\rho^2(k)}{\omega} \approx -0.05\frac{k^3}{\omega}$$

Experimentally, these phases are indeed small, but not accurately determined.

Of course the scattering contains many finer features. For example, from (18.28) and (18.32) it follows that the T matrix can be written in the form[1] $a + b\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{k}'$, where $\mathbf{k}$ and $\mathbf{k}'$ are the momenta of the incident and scattered mesons, respectively. This predicts that the recoil proton may be polarized along a direction perpendicular to the scattering plane, since the a and b terms will add or subtract, depending on the direction of the spin relative to the scattering plane. In addition, as we stated earlier, the S-wave phase shifts are not zero and become progressively more important as the energy approaches zero below the resonance energy. Above the resonance, recoil effects may be expected to complicate the scattering further, and the model that we have considered begins to make less and less sense. Nevertheless, as we have

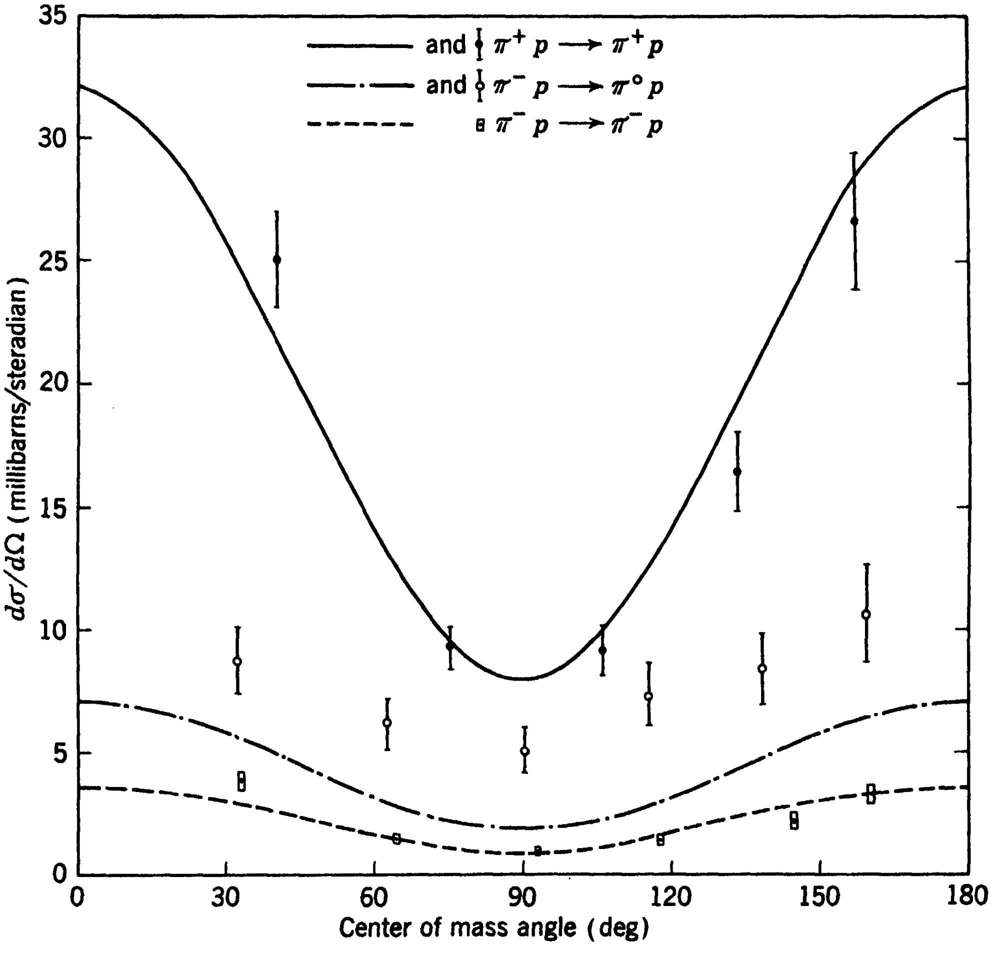

Fig. 18.6. Differential cross sections for elastic and charge-exchange scattering of 189-Mev pions by hydrogen. The experimental points are taken from H. L. Anderson, W. C. Davidon, M. Glicksman, and U. E. Kruse, *Phys. Rev.*, **100**:279 (1955). The curves are proportional to $3 \cos^2 \vartheta + 1$ and are plotted in the ratio of 9:2:1, as predicted by (18.58).

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[1] This is also the most general form allowed by parity conservation, since a and b can be functions of k^2 and $\mathbf{k} \cdot \mathbf{k}'$.

seen, the simple static model is able to account for many details of the pion-nucleon scattering over a reasonably wide range of energy.

Further Reading

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CHAPTER 19

Properties of the Nucleon

19.1. Expectation Value of the Field. The development of the last chapter allows us to gain further insight into the properties of the nucleon. The expectation values of several observables, such as the number of mesons in the cloud, their charge distribution, and their contribution to the magnetic moment of the nucleon, can be expressed in terms of the renormalized coupling constant and the scattering cross sections.

In this section we shall return to a plane-wave expansion, since most observables are more conveniently expressed by it. We consider first the quantity $a_\alpha(\mathbf{k},t)$ at $t = 0$, which is given by (15.10) or

$$\begin{aligned} a_\alpha(\mathbf{k}) &= A_\alpha(\mathbf{k}) - \int_{-\infty}^{0} dt\, e^{i\omega t} V_\alpha(\mathbf{k},t) \\ V_\alpha(\mathbf{k},t) &= f\frac{\boldsymbol{\sigma}(t)\cdot\mathbf{k}\tau_\alpha(t)\rho(\mathbf{k})}{(2\omega)^{\frac{1}{2}}(2\pi)^{\frac{3}{2}}} = V_\alpha^\dagger(\mathbf{k},t) \end{aligned} \tag{19.1}$$

The interaction $V_\alpha(\mathbf{k},t)$ is the counterpart of that defined by (18.5).

For the ground state we have $A_\alpha(\mathbf{k}) \mid \xi\rangle = 0$, and by making use of the Heisenberg time dependence of $\boldsymbol{\sigma}(t)\tau_\alpha(t)$, we find

$$\begin{aligned} \langle\xi' \mid a_\alpha(\mathbf{k}) \mid \xi\rangle &= -\frac{f_r}{(2\pi)^{\frac{3}{2}}}\frac{\rho(k)}{(2\omega)^{\frac{1}{2}}}(\xi' \mid \tau_\alpha\boldsymbol{\sigma}\cdot\mathbf{k} \mid \xi)\int_{-\infty}^{0} dt\, e^{i\omega t} \\ &= \frac{f_r}{(2\pi)^{\frac{3}{2}}}\frac{i\rho(k)}{(2\omega^3)^{\frac{1}{2}}}(\xi' \mid \tau_\alpha\boldsymbol{\sigma}\cdot\mathbf{k} \mid \xi) \end{aligned} \tag{19.2}$$

In performing the integration in (19.2), we made use of the fact that the adiabatic switching on of the field implies that

$$\int_{-\infty}^{0} dt\, e^{i\omega t} = \lim_{\substack{\alpha\geq 0 \\ \alpha\to 0}} \int_{-\infty}^{0} dt\, e^{(i\omega+\alpha)t}$$

The expectation value of the meson field becomes

$$\begin{aligned} \langle \xi' \mid \phi_\alpha(\mathbf{r}) \mid \xi \rangle &= f_r \int \frac{\rho(k)}{(2\omega^3)^{\frac{1}{2}}} (\xi' \mid i\tau_\alpha \boldsymbol{\sigma} \cdot \mathbf{k} \mid \xi) \frac{e^{i\mathbf{k}\cdot\mathbf{r}} - e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\omega)^{\frac{1}{2}}} \frac{d^3k}{8\pi^3} \\ &= f_r(\xi' \mid \tau_\alpha \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \mid \xi) Y(r) \qquad (19.3) \\ Y(r) &= \int d^3r' \, \rho(r') \frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{4\pi \, |\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

Hence, we get the result of perturbation theory, except that f is replaced by f_r. Although ϕ is not directly measurable, this result is instructive, since it is the quantity (19.3) which is connected with the large-distance (e.g., one-meson-exchange) part of the nuclear forces (see Chap. 21) and also with zero-energy πN scattering.

19.2. Ground-state Expectation Value of Observables. In order to study observables quadratic in the ϕ's, we use a complete set of eigenstates $| \text{out}, n \rangle$ of H to derive

$$\begin{aligned} \langle \xi' \mid a_{\alpha'}^\dagger(\mathbf{k}')a_\alpha(\mathbf{k}) \mid \xi \rangle &= \int_{-\infty}^{0} dt \, dt' \, e^{i(\omega t - \omega' t')} \langle \xi' \mid V_{\alpha'}^\dagger(\mathbf{k}',t')V_\alpha(\mathbf{k},t) \mid \xi \rangle \\ &= \frac{f^2\rho(k)\rho(k')}{(2\pi)^3 2(\omega\omega')^{\frac{1}{2}}} \mathop{\mathrm{S}}_n \frac{\langle \xi' \mid \tau_{\alpha'}\boldsymbol{\sigma} \cdot \mathbf{k}' \mid \text{out}, n \rangle \langle \text{out}, n \mid \tau_\alpha \boldsymbol{\sigma} \cdot \mathbf{k} \mid \xi \rangle}{(E_n + \omega)(E_n + \omega')} \qquad (19.4) \end{aligned}$$

We have again made use of the Heisenberg-operator time dependence in obtaining (19.4).

When the intermediate state $| \text{out}, n \rangle$ is the ground state, the matrix elements in (19.4) can be expressed immediately in terms of f_r. For other intermediate states the expression on the right-hand side can be related to the total cross section. This can be done with (18.11) and (18.39),

$$\begin{aligned} -2 \operatorname{Im} T_{\xi'K',\xi K} &= \mathop{\mathrm{S}}_n T_{n,\xi'K'}^* T_{n,\xi K} 2\pi\delta(\omega_n - \omega) \\ &= 2\pi \mathop{\mathrm{S}'}_n \langle \xi' \mid V_{K'} \mid \text{out}, n \rangle \langle \text{out}, n \mid V_K \mid \xi \rangle \, \delta(\omega_n - \omega) \\ &= \sum_u \frac{\mathfrak{P}_{\xi'K',\xi K}^{(u)} \sigma^{(u)}(\omega) k}{6\pi^2\omega} \qquad (19.5) \end{aligned}$$

where the sum S' is over all states excluding the ground state. In order to use (19.5), we have to transform the projection operators into our plane-wave representation. This is, in a more explicit notation,

$$\mathfrak{P}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}} = (\mathbf{k}' \mid m')\mathfrak{P}_{\xi'\alpha'm',\xi\alpha m}(m \mid \mathbf{k})$$

where

$$(m \mid \mathbf{k}) = Y_1^m = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \frac{k^m}{k}$$

This gives us

$$\mathfrak{P}^{(1)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}} = \frac{1}{12\pi kk'}(\xi'\,|\,\boldsymbol{\sigma}\cdot\mathbf{k}'\tau_{\alpha'}\boldsymbol{\sigma}\cdot\mathbf{k}\tau_{\alpha}\,|\,\xi)$$

$$\mathfrak{P}^{(2)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}} = \frac{1}{12\pi kk'}(\xi'\,|\,3\tau_{\alpha'}\tau_{\alpha}\mathbf{k}\cdot\mathbf{k}' - 2\tau_{\alpha'}\tau_{\alpha}\boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k} + 3\delta_{\alpha\alpha'}\boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k}\,|\,\xi) \tag{19.5a}$$

$$\mathfrak{P}^{(3)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}} = \frac{1}{12\pi kk'}(\xi'\,|\,(3\delta_{\alpha\alpha'} - \tau_{\alpha'}\tau_{\alpha})(3\mathbf{k}\cdot\mathbf{k}' - \boldsymbol{\sigma}\cdot\mathbf{k}'\boldsymbol{\sigma}\cdot\mathbf{k})\,|\,\xi)$$

Equation (19.5) can be written in the form

$$\underset{n}{\mathrm{S}}\,\langle\xi'\,|\,\tau_{\alpha'}\boldsymbol{\sigma}\cdot\mathbf{n}'\,|\,\text{out},n\rangle\langle\text{out},n\,|\,\tau_{\alpha}\boldsymbol{\sigma}\cdot\mathbf{n}\,|\,\xi\rangle\,\delta(\omega_n-\omega_q) = \frac{4}{3f^2q\rho^2(q)}\sum_u \mathfrak{P}^{(u)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}}\sigma_u(\omega_q) \tag{19.5b}$$

where **n** and **n**′ are unit vectors in the directions of **k** and **k**′. Multiplying (19.5*b*) with $\dfrac{1}{(\omega+\omega_q)}\dfrac{1}{(\omega'+\omega_q)}$ and integrating over ω_q, we get

$$(\xi'\,|\,a^{\dagger}_{\alpha'}(\mathbf{k}')a_{\alpha}(\mathbf{k})\,|\,\xi) = \frac{\rho(k)\rho(k')}{16\pi^3(\omega\omega')^{\frac{1}{2}}}\left[\frac{f_r^2}{\omega\omega'}(\xi'\,|\,\boldsymbol{\sigma}\cdot\mathbf{k}'\tau_{\alpha'}\boldsymbol{\sigma}\cdot\mathbf{k}\tau_{\alpha}\,|\,\xi) + \frac{4}{3}\sum_u\int_1^{\infty}d\omega_q\frac{\mathfrak{P}^{(u)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}}\sigma_u(\omega_q)}{q\rho^2(q)(\omega+\omega_q)(\omega'+\omega_q)}\right] \tag{19.6}$$

As a first application of (19.6), we calculate the mean number of virtual mesons around the nucleon. With (19.6) we find

$$\begin{aligned}\langle N\rangle &= \sum_{\alpha}\int d^3k\,\langle\xi\,|\,a^{\dagger}_{\alpha}(\mathbf{k})a_{\alpha}(\mathbf{k})\,|\,\xi\rangle \\ &= \int\frac{d^3k}{(2\pi)^3}\frac{\rho^2(k)k^2}{2\omega}\left[\frac{3f_r^2}{\omega^2} + \frac{1}{3\pi}\int_1^{\infty}\frac{d\omega_q}{q\rho^2(q)}\frac{\sigma_1(\omega_q)+4\sigma_2(\omega_q)+4\sigma_3(\omega_q)}{(\omega_q+\omega)^2}\right] \\ &= \frac{3}{\pi}\int_0^{\infty}\frac{dk\,k^4\rho^2(k)}{\omega}\left[\frac{f_r^2}{4\pi\omega^2} + \frac{1}{36\pi^2}\int_1^{\infty}\frac{d\omega_q}{q\rho^2(q)}\frac{\sigma_1(\omega_q)+4\sigma_2(\omega_q)+4\sigma_3(\omega_q)}{(\omega_q+\omega)^2}\right]\end{aligned} \tag{19.7}$$

No summation over $|\,\xi\rangle$ is implied in (19.7), and $\langle N\rangle$ is independent of the state $|\,\xi\rangle$. The first term is the renormalized Born approximation,[1] which differs from our familiar expression $\sim f^2(d^3k/\omega^3)$ of Part two by the factor k^2, which arises because we are dealing with *P*-wave mesons. Since our method always refers to the physical particle, f_r appears rather than f. The integral $\int d\omega_q\cdots$ represents contributions of higher order in f_r^2.

---

[1] We shall abbreviate this by RBA henceforth.

In exactly the same way as above, we obtain

$$\begin{aligned}\langle H_0\rangle &= \sum_\alpha \int d^3k\, \omega \langle \xi \mid a_\alpha^\dagger(\mathbf{k}) a_\alpha(\mathbf{k}) \mid \xi\rangle \\ &= \frac{3}{\pi}\int_0^\infty dk\; k^4\rho^2(k)\left[\frac{f_r^2}{4\pi\omega^2} + \frac{1}{36\pi^2}\int_1^\infty \frac{d\omega_q}{q\rho^2(q)}\right. \\ &\qquad \left.\times\, \frac{\sigma_1(\omega_q) + 4\sigma_2(\omega_q) + 4\sigma_3(\omega_q)}{(\omega+\omega_q)^2}\right]. \end{aligned} \tag{19.8}$$

and, with appropriate changes,

$$\begin{aligned}\langle H'\rangle &= \sum_\alpha \int d^3k\, \langle \xi \mid a_\alpha(\mathbf{k}) i V_\alpha(\mathbf{k}) - i a_\alpha^\dagger(\mathbf{k}) V_\alpha^\dagger(\mathbf{k}) \mid \xi\rangle \\ &= \mathop{\mathrm{S}}_n \sum_\alpha \int d^3k\, \frac{2\langle \xi \mid V_\alpha(\mathbf{k}) \mid \text{out}, n\rangle\langle \text{out}, n \mid V_\alpha(\mathbf{k}) \mid \xi\rangle}{\omega + E_n} \\ &= -\frac{6}{\pi}\int_0^\infty \frac{dk\, k^4\rho^2(k)}{\omega}\left[\frac{f_r^2}{4\pi\omega} + \frac{1}{36\pi^2}\int_1^\infty \frac{d\omega_q}{q\rho^2(q)}\, \frac{\sigma_1(\omega_q) + 4\sigma_2(\omega_q) + 4\sigma_3(\omega_q)}{(\omega+\omega_q)}\right] \end{aligned} \tag{19.9}$$

Comparing (19.8) and (19.9), we see that the Born-approximation term of $\langle H_0\rangle$ is half that of $-\langle H'\rangle$ but that the double integral in H_0 is less than half that of H'. Hence, we have the "virial theorem"

$$\langle H_0\rangle \le -\tfrac{1}{2}\langle H'\rangle \qquad \text{and} \qquad \langle \mathscr{E}_0\rangle \le -\langle H_0\rangle \tag{19.10}$$

which means that the interaction H' makes the nucleon dynamically lighter.

19.3. Renormalization Constants and Other Parameters of the Static Model. To learn more about $\mathscr{E}_0$, we must first digress to discuss the parameters $\mathfrak{r}_1$, $\mathfrak{r}_2$, f^2, $\omega_{\max}$, which are relevant to the static-model description of the nucleon. Some of the relations we need are obtained by integrating (19.5*b*) over ω_n:

$$\begin{aligned}&\langle \xi' \mid (\tau\sigma)_{K'}(\tau\sigma)_K \mid \xi\rangle - \langle \xi' \mid (\tau\sigma)_{K'} \mid \xi''\rangle\langle \xi'' \mid (\tau\sigma)_K \mid \xi\rangle \\ &\qquad\qquad = \frac{4}{3f^2}\int_1^\infty \frac{d\omega_n}{k_n\rho^2(k_n)} \sum_u \mathfrak{P}^{(u)}_{\xi'K',\xi K}\sigma_u(\omega_n)\end{aligned} \tag{19.11}$$

The expectation value can be worked out with (17.5), (17.3), and (18.32). For example,

$$\begin{aligned}\langle \xi' \mid (\tau\sigma)_{K'}(\tau\sigma)_K \mid \xi\rangle = (\xi' \mid \delta_{KK'} &- (\boldsymbol{\sigma}\cdot \boldsymbol{l}_{K'K} + \boldsymbol{\tau}\cdot \mathbf{t}_{K'K})\,\mathfrak{r}_1 \\ &+ \boldsymbol{\sigma}\cdot \boldsymbol{l}_{K'K}\mathbf{t}_{K'K}\cdot\boldsymbol{\tau}\mathfrak{r}_2 \mid \xi)\end{aligned} \tag{19.12}$$

To cast the integral in (19.11) into a comparable form, we use (19.5*b*):

$$\sum_u \mathfrak{P}^{(u)}_{\xi' K',\xi K} h^{(u)} = (\xi' \mid \delta_{KK'}(h_1 + 4h_2 + 4h_3) - (\boldsymbol{\sigma} \cdot \boldsymbol{l}_{K'K} + \boldsymbol{\tau} \cdot \mathbf{t}_{K'K}) \times (h_1 + h_2 - 2h_3) + \boldsymbol{\sigma} \cdot \boldsymbol{l}_{K'K} \boldsymbol{\tau} \cdot \mathbf{t}_{K'K}(h_1 - 2h_2 + h_3) \mid \xi) \frac{1}{12\pi} \tag{19.13}$$

Collecting these results and equating the coefficients of 1, $\boldsymbol{\sigma} \cdot \boldsymbol{l} + \boldsymbol{\tau} \cdot \mathbf{t}$ and $\boldsymbol{\sigma} \cdot \boldsymbol{l}\, \boldsymbol{\tau} \cdot \mathbf{t}$, we get

$$\frac{1}{36\pi^2} \int \frac{d\omega_n}{k_n \rho^2(k_n)} \begin{Bmatrix} \sigma_1 + 4\sigma_2 + 4\sigma_3 \\ \sigma_1 + \sigma_2 - 2\sigma_3 \\ \sigma_1 - 2\sigma_2 + \sigma_3 \end{Bmatrix} = \frac{f^2}{4\pi} \begin{Bmatrix} 1 - \mathfrak{r}_2^2 \\ \mathfrak{r}_1 - \mathfrak{r}_2^2 \\ \mathfrak{r}_2 - \mathfrak{r}_2^2 \end{Bmatrix} \qquad \begin{matrix} (19.14a) \\ (19.14b) \\ (19.14c) \end{matrix}$$

In principle, these relations can be used to calculate the unrenormalized coupling constant and the renormalization constants from observable quantities. In practice, the difficulty arises that in (19.14) there must be significant contributions to the integrals from high energies, where the model would not be expected to describe physical reality. For instance, taking the combinations $(a + 2b - 3c)$ of (19.14), we find

$$\int \frac{d\omega_n}{k_n \rho^2(k_n)} \sigma_2 \geq \frac{1}{4} \int \frac{d\omega_n}{k_n \rho^2(k_n)} \sigma_3 \tag{19.15}$$

since we know that $(1 + 2\mathfrak{r}_1 - 3\mathfrak{r}_2) \geq 0$ from (17.13). Experimentally, σ_2 is only a few per cent of σ_3 up to 300 Mev, so that $\int \sigma_2\, d\omega_n / [k_n \rho^2(k_n)]$ must get large contributions from higher energies. This difficulty does not occur in the Low equations (of Chap. 18) for pion-nucleon scattering or in the expression for $\langle N \rangle$, where the high-energy contributions to the integral are suppressed by one or more extra powers of ω in the denominator.

Nevertheless, if we insert the low-energy-scattering data into the relations we obtained, or shall obtain,[1] and leave the high-energy behavior open, we can get a fairly good idea of what the various parameters must be if the theory is to agree with experiments at low energies. We shall not go into this analysis[2] but shall only quote that for

$$\begin{array}{llll} \dfrac{f_r^2}{4\pi} = 0.08 & \text{and} & \rho(k) = \dfrac{\zeta^2}{k^2 + \zeta^2} & \rho(r) \propto \dfrac{e^{-\zeta r}}{r} \\ \dfrac{f^2}{4\pi} = 0.22 & \mathfrak{r}_1 = 0.37 & & \\ \zeta = 4.7 & \mathfrak{r}_2 = 0.58 & & \end{array} \tag{19.16}$$

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[1] The data below make use of the meson's charge contribution to the nucleon, which we shall discuss shortly.

[2] See S. Fubini and W. Thirring, *Phys. Rev.*, **105**:1382 (1957).

the various relations and inequalities can be satisfied. Of course, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are not independent and, in principle, could be calculated from f and ρ. The discussion given in Chap. 17 shows that it is not inconceivable that the renormalization constants $\mathfrak{r}_1$ and $\mathfrak{r}_2$ have the given numerical value, if ρ and f_r are chosen as in (19.16). For the physical significance of $\mathfrak{r}_1$ and $\mathfrak{r}_2$ in terms of probabilities, we have to refer back to Chap. 17. From $\mathfrak{r}_1 \approx 0.4$ we find, for instance, that the probability of finding a bare proton (neutron) in the physical proton is 70 per cent (30 per cent). From $\mathfrak{r}_2 \approx 0.5$ we deduce that the probability of finding a bare proton with spin up, or a bare neutron with spin down, in a physical proton with $J_z = \frac{1}{2}$ is 75 per cent.

19.4. Nucleon Self-energy. With the cutoff (19.16) we can also compute the expectation values (19.7) to (19.9). In these expectations the denominators in the integral over $d\omega_n$ are sufficiently large that the main contribution to this integral arises from lower energies where the $\frac{3}{2},\frac{3}{2}$-resonance dominates the scattering. We, therefore, neglect σ_1 and σ_2, and for σ_3 we insert two-thirds of the measured π^+p total cross section. The factor of $\frac{2}{3}$ arises as follows. Whereas the π^+p system is always in a $T = \frac{3}{2}$ state, it can be in either a $J = \frac{3}{2}$ state or a $J = \frac{1}{2}$ state. The probability for the former is $\frac{2}{3}$ [see (16.37)]. With the above approximation the remaining k integration is elementary, and we find

$$\langle N \rangle \approx 1 \tag{19.17a}$$

$$\langle H_0 \rangle \approx 8 \qquad \langle H_0 + H' \rangle = \mathscr{E}_0 = -11 \tag{19.17b}$$

The major contribution to (19.17) arises from the RBA terms, whereas the correction terms amount to about 20 per cent. Because the integral over d^3k has a k^4 term in the numerator, there are large contributions from energies close to $\omega_{\max}$, in particular to (19.17*b*). The numbers should, therefore, not be taken too seriously, because the static model is of doubtful validity at energies $\sim\omega_{\max}$. Nevertheless, the orders of magnitude, at least of $\langle N \rangle$, should be correct. That this is the case can be inferred, for example, from the experimental data of proton-antiproton annihilation, where we find that the average number of pions produced is ~5.[1] We are tempted to picture this event as follows. The nucleon cores annihilate, giving rise to the minimum number of pions, which is two (assuming energy-momentum conservation) or three. The other two or three mesons are then supplied by the meson clouds of the two heavy particles. This number compares favorably with the mean number of mesons evaluated above.

Returning to our expression for $\mathscr{E}_0$ [see (19.8) and (19.9)], we can derive another instructive inequality. First, we note that all the correction terms (under the integral $\int d\omega_n$) contribute with the same sign. Furthermore, for all physical energies ω_n, we have

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[1] E. Segré, *Ann. Rev. Nuclear Sci.*, **8**:127 (1958) (see especially pp. 148–149).

$$0 \leq \frac{2}{\omega(\omega + \omega_n)} - \frac{1}{(\omega + \omega_n)^2} \leq \frac{1}{\omega^2} \tag{19.18}$$

which, together with (19.14*a*), tells us that

$$\frac{f_r^2}{4\pi}\frac{3}{\pi}\int_0^\infty \frac{dk\, \rho^2(k)k^4}{\omega^2} \leq \left|\mathscr{E}_0\right| \leq \frac{f^2}{4\pi}\frac{3}{\pi}\int_0^\infty \frac{dk\, \rho^2(k)k^4}{\omega^2} \tag{19.19}$$

Thus, the exact $\mathscr{E}_0$ lies between the Born approximations taken with the renormalized and unrenormalized coupling constants, respectively. In the strong-coupling (or classical) limit, where $f^2 = 9f_r^2$, we had, from (17.63),

$$\mathscr{E}_0 = -\frac{f^2}{4\pi}\frac{1}{\pi}\int_0^\infty \frac{dk\, \rho^2(k)k^4}{\omega^2} = -\frac{9f_r^2}{4\pi}\frac{1}{\pi}\int_0^\infty \frac{\rho^2(k)k^4\, dk}{\omega^2} \tag{19.20}$$

which is the geometric mean between the two limits.

19.5. Charge and Current Distribution of Physical Nucleon. Magnetic Moment. Finally, we shall evaluate the charge and current distribution associated with the meson cloud and the resulting magnetic moment. These quantities can be measured in detail (e.g., by means of high-energy electron-nucleon scattering) and are, therefore, of special interest. To this end we need the expectation values of $\langle \xi' \mid a_{\alpha'}(\mathbf{k}')a_\alpha(\mathbf{k}) \mid \xi \rangle$, e.g., the amplitude for finding pairs in the nucleon, which we derive with the aid of the equation

$$\begin{aligned} \frac{\partial}{\partial t}\left[a_{\alpha'}(\mathbf{k}',t)a_\alpha(\mathbf{k},t)\right]_{t=0} &= i\left[H, a_{\alpha'}(\mathbf{k}')a_\alpha(\mathbf{k})\right] \\ &= -i(\omega + \omega')a_{\alpha'}(\mathbf{k}')a_\alpha(\mathbf{k}) - a_{\alpha'}(\mathbf{k}')V_{\alpha'}(\mathbf{k}') - a_\alpha(\mathbf{k})V_\alpha(\mathbf{k}) \end{aligned} \tag{19.21}$$

Remembering that expectation values taken between stationary states are constant, since $\langle \xi \mid [H,\mathcal{O}] \mid \xi \rangle = 0$, and that a and V commute, we find, by means of (19.1),

$$\begin{aligned} \langle \xi' \mid a_{\alpha'}(\mathbf{k}')a_\alpha(\mathbf{k}) \mid \xi \rangle &= i\,\frac{\langle \xi' \mid V_{\alpha'}(\mathbf{k}')a_{\alpha'}(\mathbf{k}') + V_\alpha(\mathbf{k})a_\alpha(\mathbf{k}) \mid \xi \rangle}{\omega + \omega'} \\ &= \frac{-1}{\omega + \omega'}\mathop{\mathrm{S}}_n\left[\frac{\langle \xi' \mid V_{\alpha'}(\mathbf{k}') \mid \text{out}, n\rangle\langle \text{out}, n \mid V_\alpha(\mathbf{k}) \mid \xi \rangle}{\omega_n + \omega}\right. \\ &\qquad \left. + \frac{\langle \xi' \mid V_\alpha(\mathbf{k}) \mid \text{out}, n\rangle\langle \text{out}, n \mid V_{\alpha'}(\mathbf{k}') \mid \xi \rangle}{\omega_n + \omega'}\right] \end{aligned} \tag{19.22}$$

and similarly, by hermitian conjugation,

$$\begin{aligned}\langle\xi'\mid a^{\dagger}_{\alpha'}(\mathbf{k}')a^{\dagger}_{\alpha}(\mathbf{k})\mid\xi\rangle &= -i\,\frac{\langle\xi'\mid a^{\dagger}_{\alpha'}(\mathbf{k}')V_{\alpha'}(\mathbf{k}')+a^{\dagger}_{\alpha}(\mathbf{k})V_{\alpha}(\mathbf{k})\mid\xi\rangle}{\omega+\omega'}\\ &= \frac{-1}{\omega+\omega'}\mathop{\mathsf{S}}_{n}\left[\frac{\langle\xi'\mid V_{\alpha'}(\mathbf{k}')\mid\text{out},n\rangle\langle\text{out},n\mid V_{\alpha}(\mathbf{k})\mid\xi\rangle}{\omega_n+\omega'}\right.\\ &\qquad\left.+\frac{\langle\xi'\mid V_{\alpha}(\mathbf{k})\mid\text{out},n\rangle\langle\text{out},n\mid V_{\alpha'}(\mathbf{k}')\mid\xi\rangle}{\omega_n+\omega}\right]\end{aligned}$$

More specifically, using the results of Sec. 19.2 and the notation and techniques developed there, we have the result

$$\begin{aligned}\langle\xi'\mid a_{\alpha'}(\mathbf{k}')a_{\alpha}(\mathbf{k})\mid\xi\rangle &= -\frac{\rho(k)\rho(k')}{8\pi^3(4\omega\omega')^{\frac12}(\omega+\omega')}\\ &\quad\times\left\{\left(\xi'\left|\frac{\tau_{\alpha'}(\boldsymbol{\sigma}\cdot{}')\tau_{\alpha}(\boldsymbol{\sigma}\cdot\mathbf{k})}{\omega}+\frac{\tau_{\alpha}(\boldsymbol{\sigma}\cdot\mathbf{k})\tau_{\alpha'}(\boldsymbol{\sigma}\cdot\mathbf{k}')}{\omega'}\right|\xi\right)\right.\\ &\quad\left.\times f_r^2+\frac{4}{3}\int_1^{\infty}\frac{d\omega_n}{k_n\rho^2(k_n)}\sum_u\sigma_u\left[\frac{\mathfrak{P}^{(u)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}}}{\omega_n+\omega}+\frac{\mathfrak{P}^{(u)}_{\xi'\alpha\mathbf{k},\xi\alpha'\mathbf{k}'}}{\omega_n+\omega'}\right]\right\}\end{aligned}\tag{19.23}$$

$$\begin{aligned}\langle\xi'\mid a^{\dagger}_{\alpha'}(\mathbf{k}')a^{\dagger}_{\alpha}(\mathbf{k})\mid\xi\rangle &= -\frac{\rho(k)\rho(k')}{(4\omega\omega')^{\frac12}(\omega+\omega')}\frac{1}{8\pi^3}\\ &\quad\times\left\{\left(\xi'\left|\frac{\tau_{\alpha'}(\boldsymbol{\sigma}\cdot\mathbf{k}')\tau_{\alpha}(\boldsymbol{\sigma}\cdot\mathbf{k})}{\omega'}+\frac{\tau_{\alpha}(\boldsymbol{\sigma}\cdot\mathbf{k})\tau_{\alpha'}(\boldsymbol{\sigma}\cdot\mathbf{k}')}{\omega}\right|\xi\right)\right.\\ &\quad\left.\times f_r^2+\frac{4}{3}\int_1^{\infty}\frac{d\omega_n}{k_n\rho^2(k_n)}\sum_u\sigma_u\left[\frac{\mathfrak{P}^{(u)}_{\xi'\alpha'\mathbf{k}',\xi\alpha\mathbf{k}}}{\omega_n+\omega'}+\frac{\mathfrak{P}^{(u)}_{\xi'\alpha\mathbf{k},\xi\alpha'\mathbf{k}'}}{\omega_n+\omega}\right]\right\}\end{aligned}$$

With these expressions we are able to calculate local quantities quadratic in ϕ. For the current (7.12), $\mathbf{j}_\pi = -e(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)$, we find

$$\begin{aligned}&\langle\xi'\mid j_{\pi j}(\mathbf{r})\mid\xi\rangle\\ &= -e\mathop{\mathsf{S}}_{\mathbf{k},\mathbf{k}'}\frac{e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}}{(4\omega\omega')^{\frac12}}\,i(k_j-k_j')\\ &\qquad\times\langle\xi'\mid a_1(\mathbf{k}')a_2(\mathbf{k})+a_1^{\dagger}(-\mathbf{k}')a_2(\mathbf{k})+a_1(\mathbf{k}')a_2^{\dagger}(-\mathbf{k})+a_1^{\dagger}(-\mathbf{k}')a_2^{\dagger}(-\mathbf{k})\mid\xi\rangle\\ &= e\mathop{\mathsf{S}}_{\mathbf{k},\mathbf{k}'}\frac{e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\rho(k)\rho(k')}{\omega\omega'}\,i(k_j-k_j')(\xi'\mid\tau_3\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{k}'\mid\xi)\\ &\qquad\times\left[\frac{f_r^2}{\omega\omega'}+\frac{1}{9\pi}\int_1^{\infty}\frac{d\omega_n}{k_n\rho^2(k_n)}\frac{\sigma_1-2\sigma_2+\sigma_3}{(\omega_n+\omega)(\omega_n+\omega')}\left(1+\frac{\omega_n}{\omega+\omega'}\right)\right]\end{aligned}\tag{19.24}$$

where we have used

$$\tau_1\boldsymbol{\sigma}\cdot\mathbf{k}\tau_2\boldsymbol{\sigma}\cdot\mathbf{k}'+\tau_2\boldsymbol{\sigma}\cdot\mathbf{k}'\tau_1\boldsymbol{\sigma}\cdot\mathbf{k} = -2\tau_3\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{k}'\tag{19.25}$$

We can further reduce (19.24) by carrying out the angular integrations. We note that

$$\int d^3k\, d^3k'\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} i(\mathbf{k}-\mathbf{k}')(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{k}')f(\omega,\omega')$$
$$= \nabla\times\boldsymbol{\sigma}\int d^3k\, d^3k'\, \frac{k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')^2}{(\mathbf{k}+\mathbf{k}')^2}\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} f(\omega,\omega')$$

This can be most easily seen by considering the j-component and rewriting the integral as

$$\tfrac{1}{2}\sum_i(\nabla\times\boldsymbol{\sigma})_i\int d^3k\, d^3k'\,(k_i-k_i')(k_j-k_j')\,f(\omega,\omega')\,e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}$$

If we make a change of variables to $\frac{1}{2}(\mathbf{k}-\mathbf{k}')=\boldsymbol{\kappa}$ and $(\mathbf{k}+\mathbf{k}')=\mathbf{K}$, the integral becomes

$$2\sum_i(\nabla\times\boldsymbol{\sigma})_i\int d^3\kappa\, d^3K\,\kappa_i\kappa_j\,f(\omega,\omega')\,e^{i\mathbf{K}\cdot\mathbf{r}}$$
$$=\sum_i(\nabla\times\boldsymbol{\sigma})_i\int d^3K\,[F_1(K^2)\,\delta_{ij}+F_2(K^2)K_iK_j]\,e^{i\mathbf{K}\cdot\mathbf{r}}$$

where F_1 and F_2 are arbitrary functions. The F_2 term will not contribute because $(\nabla\times\boldsymbol{\sigma})$ operating on it gives zero. For F_1 we find

$$F_1=\int d^3\kappa\left[\kappa^2-\frac{(\boldsymbol{\kappa}\cdot\mathbf{K})^2}{K^2}\right]f(\omega,\omega')$$

We can therefore rewrite (19.24) as

$$\langle\xi\mid\mathbf{j}_\pi(\mathbf{r})\mid\xi\rangle$$
$$= e\,\nabla\times(\xi\mid\tau_3\boldsymbol{\sigma}\mid\xi)\int\frac{d^3k\,d^3k'}{(2\pi)^6}\,\frac{\rho(k)\rho(k')}{\omega\omega'}\,\frac{k^2k'^2-(\mathbf{k}\cdot\mathbf{k}')^2}{(\mathbf{k}+\mathbf{k}')^2}\,e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}$$
$$\times\left[\frac{f_r^2}{\omega\omega'}+\frac{1}{9\pi}\int_1^\infty\frac{d\omega_n}{k_n\rho^2(k_n)}\,\frac{\sigma_1-2\sigma_2+\sigma_3}{(\omega_n+\omega)(\omega_n+\omega')}\left(1+\frac{\omega_n}{\omega+\omega'}\right)\right] \tag{19.26}$$

Since the integral in (19.26) depends only on $|\mathbf{r}|$, the current of the meson field is perpendicular to $\boldsymbol{\sigma}$ and $\mathbf{r}$ corresponding to a magnetization density $\boldsymbol{\sigma}f(r)$ (see Fig. 19.1). It is positive for the proton and equal but opposite for the neutron. With (19.16) we find that about 70 per cent of the current comes from the RBA term $\propto\langle\xi\mid\phi\mid\xi''\rangle\langle\xi''\mid\nabla\phi\mid\xi\rangle$ which gives a contribution at large distances $\propto\boldsymbol{\sigma}\times\nabla e^{-2r}/r^2$. This is in qualitative agreement with the results from electron-scattering experiments,[1] which show that the current which creates the magnetic moment of the proton and neutron is spread out over a region with mean-square radius $\sim 0.7 \times 10^{-13}$ cm $\approx \frac{1}{2}$.

Similar behavior characterizes the charge distribution, which can be calculated along the same lines,[2]

$$\langle \xi \mid Q_\pi(\mathbf{r}) \mid \xi \rangle = -e(\phi_2\dot{\phi}_1 - \phi_1\dot{\phi}_2)$$

$$= \frac{ie}{2} \mathop{\mathrm{S}}_{\mathbf{k},\mathbf{k}'} (2\pi)^3 \frac{e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}}{(\omega\omega')^{\frac{1}{2}}} \langle \xi \mid (\omega - \omega')[a_1(\mathbf{k})a_2(\mathbf{k}') - a_1^\dagger(-\mathbf{k})a_2^\dagger(-\mathbf{k}')]$$

$$- (\omega + \omega')[a_1^\dagger(-\mathbf{k})a_2(\mathbf{k}') - a_2^\dagger(-\mathbf{k}')a_1(\mathbf{k})] \mid \xi \rangle \qquad (19.27a)$$

$$\langle \xi \mid Q_\pi(\mathbf{r}) \mid \xi \rangle = -2e(\xi \mid \tau_3 \mid \xi) \mathop{\mathrm{S}}_{\mathbf{k},\mathbf{k}'} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \frac{\rho(k)\rho(k')\mathbf{k}\cdot\mathbf{k}'}{\omega + \omega'}$$

$$\times \left[\frac{f_r^2}{\omega\omega'} + \frac{1}{9\pi}\int_1^\infty \frac{d\omega_n}{k_n\rho^2(k_n)} \frac{\sigma_1 + \sigma_2 - 2\sigma_3}{(\omega_n + \omega)(\omega_n + \omega')}\right] \qquad (19.27b)$$

since
$$\tau_1\boldsymbol{\sigma}\cdot\mathbf{k}\tau_2\boldsymbol{\sigma}\cdot\mathbf{k}' - \tau_2\boldsymbol{\sigma}\cdot\mathbf{k}'\tau_1\boldsymbol{\sigma}\cdot\mathbf{k} = 2i\tau_3\mathbf{k}\cdot\mathbf{k}' \qquad (19.28)$$

The charge density $\langle \mid Q_\pi(\mathbf{r}) \mid \rangle$ is also equal but opposite in sign for a proton and neutron, the dominant RBA term behaving like e^{-2r}/r^2 asymptotically. The equal but opposite-in-sign charge density agrees with a naïve picture, in which the virtual processes $p \rightarrow n + \pi^+$ and $n \rightarrow p + \pi^-$ create a positive cloud around the proton and a negative cloud around the neutron. Hence, the neutron, though neutral as a whole, has some electric structure. It is, in this respect, like a hydrogen atom, positive inside and negative outside.

To get the total charge, we have to add the charge of the bare nucleon to the charge of the cloud. It seems reasonable to assume that the former has a distribution $\propto \rho(r)$ and, hence, will be

$$e\langle \xi \mid \frac{1 + \tau_3}{2} \rho(r) \mid \xi \rangle$$

With $\mathbf{r}_1$ given by (19.16), we obtain $0.7\rho(r)$ for the physical proton and

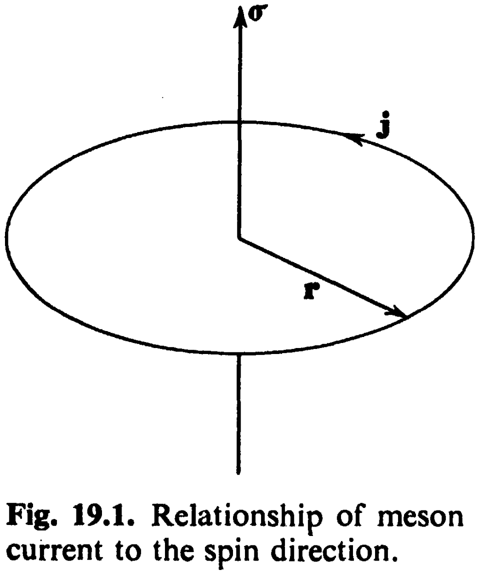

Fig. 19.1. Relationship of meson current to the spin direction.

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[1] R. Hofstadter, F. Bumiller, and M. R. Yearian, *Revs. Modern Phys.*, **30**:482 (1958).

[2] The terms proportional to $a_1(\mathbf{k}')V_2(\mathbf{k})$, $V_1(\mathbf{k}')a_2(\mathbf{k})$, etc., which arise from $a_1\dot{a}_2$ $\dot{a}_1a_2$, etc., cancel in the expression (19.27*a*) for $Q(\mathbf{r})$.

$0.3\rho(r)$ for the physical neutron. Hence, the anticipated charge distribution of the nucleons, including the charge of the bare nucleon, should have a radial dependence like that shown in Fig. 19.2. This is not borne out by electron-scattering experiments.[1] Whereas the mean-square radius

$$\langle r^2 \rangle = e^{-1} \int Q(\mathbf{r}) r^2 \, d^3r$$

of the charge distribution of the proton is 0.75×10^{-13} cm, as expected, that of the neutron is much less, if not zero. The most plausible answer to this puzzle at present is that the charge of the pion itself is spread out over a distance of about $\frac{1}{4}$.[2] Averaging the neutron charge distribution over this distance will considerably reduce the mean-square radius, whereas that of the proton will not be affected materially.

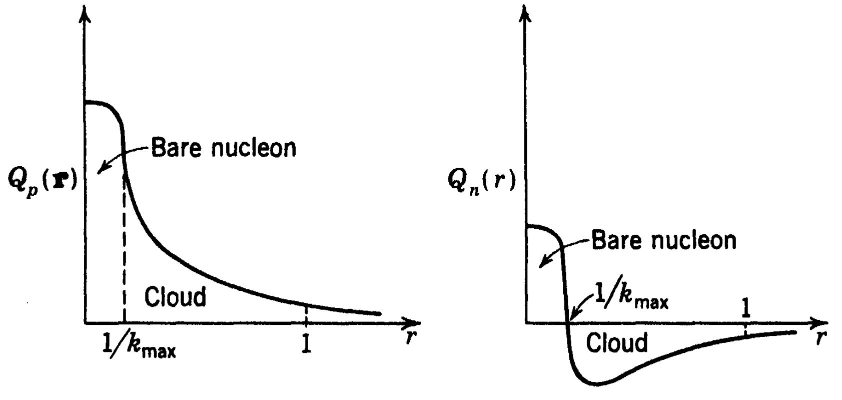

Fig. 19.2. Static charge distributions of the physical nucleons.

The integrated charge and the total magnetic moment of the pion cloud can be directly obtained from our results. For the former we find

$$Q_\pi = e(\xi \mid \tau_3 \mid \xi) \frac{2}{\pi} \int_0^\infty \frac{dk\, k^4 \rho^2(k)}{\omega} \left[\frac{f_r^2}{4\pi\omega^2} + \frac{1}{36\pi^2} \int_1^\infty \frac{d\omega_n}{k_n \rho^2(k)} \frac{\sigma_1 + \sigma_2 - 2\sigma_3}{(\omega_n + \omega)^2} \right] \tag{19.29}$$

Since the integrated charge, including the nucleon contribution, is $Q = Q_\pi + Q_N$, with

$$Q_N = \langle \xi \left| \frac{1 + \tau_3}{2} \right| \xi \rangle = \left(\xi \left| \frac{1 + \tau_3 \mathfrak{r}_1}{2} \right| \xi \right)$$

and the Q of the proton or neutron state must be 1 or 0, Eq. (19.29) is another relation for $\mathfrak{r}_1$. It also gives an upper limit for $\omega_{\max}$, since the charge of the meson cloud must not exceed e. Even if we neglect all cross sections except σ_3, then for too large a cutoff $\omega_{\max}$, the integral $\int d^3k$ becomes too big. These observations were crucial in determining the values of $\omega_{\max}$ and $\mathfrak{r}_1$ quoted in Sec. 19.3. The RBA term is just $2e/3$ of the one for $\langle N \rangle$ [see (19.7)], as one would expect, since (16.34) tells us that two-thirds of the time the proton is dissociated according to $p \rightarrow n + \pi^+$ and for one-third of the time into $p + \pi^0$.

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[1] See Hofstadter, Bumiller, and Yearian, *op. cit.*

[2] This is thought to arise through a pion-pion interaction. See, e.g., W. R. Frazer and J. R. Fulco, *Phys. Rev. Letters*, **2**:365 (1959).

For the magnetic moment we find, by means of a partial integration,

$$\mathscr{M}_\pi = \frac{e}{2} \int \mathbf{r} \times \mathbf{j}\, d^3r = \boldsymbol{\sigma}\tau_3 \frac{4}{3\pi} e \int_0^\infty \frac{dk\, k^4 \rho^2(k)}{\omega^2}$$

$$\times \left[\frac{f_r^2}{4\pi\omega^2} + \frac{1}{36\pi^2} \int_1^\infty \frac{d\omega_n}{k_n \rho^2(k_n)} \frac{\sigma_1 - 2\sigma_2 + \sigma_3}{(\omega_n + \omega)^2} \left(1 + \frac{\omega_n}{2\omega} \right) \right] \tag{19.30}$$

$$\langle \xi | \mathscr{M}_\pi | \xi \rangle = (\xi | \tau_3 \boldsymbol{\sigma} | \xi) e \mathfrak{M}_\pi$$

One would expect the magnetic moment, when measured in nuclear magnetons, to be of the order of the (mean number of mesons) × (mass ratio of the nucleon to that of the meson), since the magnetic moment of a spin-zero particle with $l = 1$ is the mass ratio multiplied by the magneton of a spin-$\frac{1}{2}$ nucleon. Experimentally,

$$\mathscr{M}_p \approx 2.8 \boldsymbol{\sigma} \frac{e}{2M} \qquad \mathscr{M}_n \approx -1.9 \boldsymbol{\sigma} \frac{e}{2M}$$

so that the moment of the meson cloud is in the right direction. But to do justice to the experimental physicists who measured these quantities to an accuracy of five figures,[1] we have to do better than that. A more refined argument is as follows. For either nucleon we find from (16.34) that both the probability of the virtual mesons being charged and the probability of their having an $l_z = 1$ are $\frac{2}{3}$. Furthermore, what counts is the relativistic mass (e.g., the meson total energy ω), so that

$$\mathfrak{M} = \frac{4}{9} \frac{\bar{N}}{\omega}$$

is expected. If $\bar{N}/\omega$ is interpreted as $\int N(k)\, dk/\omega$ we obtain from the RBA of (19.7)

$$\mathfrak{M} = \frac{2}{9} \frac{3}{\pi} \int \frac{k^4 \rho^2(k)}{\omega^2} \frac{f_r^2}{4\pi\omega^2}$$

which is one-half of the RBA of (19.30). The reason is that our argument implies that in $\mathbf{j}$ we keep only the terms proportional to $a^\dagger a$. However, the pair terms aa and $a^\dagger a^\dagger$ contribute to the RBA an equal amount, so that this more sophisticated argument is wrong by a factor 2.[¶] If the contribution from $\int_1^\infty d\omega_n \cdots$ is included, then the exact result with (19.16) is

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[1] For references, see, e.g., J. W. M. DuMond, *Ann. phys.*, 7:365 (1959); and J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," p. 31, John Wiley & Sons, Inc., New York, 1952.

$$\mathscr{M} = (\tau_3 \boldsymbol{\sigma}) \times 1.3 \qquad \text{nuclear magnetons} \tag{19.31}$$

This rather small value is mainly due to the fact that most of the mesons have high energy, so that the effective mass ratio is $\sim M/\omega_{\max}$ rather than M. To discuss the significance of (19.31), we introduce the isovector and scalar parts of $\mathscr{M}$ by

$$\mathscr{M} = (\mathfrak{M}_v \tau_3 + \mathfrak{M}_s) \frac{e}{2M} \boldsymbol{\sigma}$$

where experimentally

$$\mathfrak{M}_v = 2.35 \qquad \mathfrak{M}_s = 0.45$$

The meson cloud contributes 1.3 to $\mathfrak{M}_v$, so that an extra contribution of 1 nuclear magneton for $\mathfrak{M}_v$ and $\mathfrak{M}_s$[§] must be accounted for by other phenomena. These may be the magnetic moments of the bare protons, antinucleons, heavy mesons, and hyperons, etc. They seem to give quite substantial corrections to $\mathscr{M}$. We shall see in the next chapter that the static model is unable to predict electromagnetic phenomena at short distances (inside the core), which should not surprise us. All we can say is that the first figure of the magnetic moments can be understood in terms of a simple model, but it will take a while until we can account for the other four measured figures.

Further Reading

M. Cini and S. Fubini, *Nuovo cimento*, **3**:764 (1956).
S. Fubini, *Nuovo cimento*, **3**:1425 (1956).
H. Miyazawa, *Phys. Rev.*, **101**:1564 (1956).
S. Fubini and W. Thirring, *Phys. Rev.*, **105**:1382 (1959).
G. Salzman, *Phys. Rev.*, **105**:1076 (1959).

[¶] The strong-coupling result is the correct RBA × $\frac{3}{4}$. See W. Pauli and S. M. Dancoff, *Phys. Rev.*, **62**:85 (1942).

[§] Note that $\mathfrak{M}_s = 0$ in the strong-coupling limit, since $\langle \boldsymbol{\sigma} \rangle = 0$, irrespective of the core contributions assumed.

CHAPTER 20

Electromagnetic Phenomena

20.1. Contributions to Charge and Current Operators. Electromagnetic effects are important tools for studying the pion-nucleon system. The interactions of photons with matter are well understood[1] and can, therefore, be used to give us further insight into the structure of the nucleon and its interaction with pions. Part of this study has already been carried out in the last chapter; here, we wish to extend it to meson production and to photon scattering (Compton effect).

To this end, we first must define the total charge density and current in the static model. The charge density includes contributions from mesons,[2]

$$Q_\pi(\mathbf{r}) = e(\dot{\phi}_2\phi_1 - \dot{\phi}_1\phi_2) \tag{20.1}$$

and from the bare nucleon

$$Q_N(\mathbf{r}) = e\,\frac{1+\tau_3}{2}\,\rho(r) \tag{20.2}$$

The spatial distribution of Q_N is assumed to arise from the virtual particles which make up the source.[3] Similarly, the meson current operator is

$$\mathbf{j}_\pi = e(\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2) \tag{20.3}$$

and that of the bare nucleons arises from their normal (Dirac) moment and the contribution of other virtual particles. This part should be expressible in terms of only the nucleon operators $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$; we assume that it is divergence-free and hence that

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[1] There are reservations in the static model which will appear shortly.

[2] A plane-wave description of the meson field is more useful in this chapter, because $Q_\pi(\mathbf{r})$ and $\mathbf{j}_\pi(\mathbf{r})$ will be needed in other than P states.

[3] Strictly speaking, the bare neutron is then also expected to have a bare-nucleon charge density with an average value of 0, but this is neglected.

$$\mathbf{j}_N(\mathbf{r}) = \nabla \times \boldsymbol{\mu}_N(\mathbf{r}) \tag{20.4}$$

where $\boldsymbol{\mu}_N(\mathbf{r})$ is related to part of the magnetic moment, and we shall shortly determine its form. The only feature of $\mathbf{j}_N$ which we shall need before we define the operator more carefully is that it does not depend on meson-creation and -destruction operators, so that

$$[a_\alpha(\mathbf{k}),\mathbf{j}_N] = [a_\alpha^\dagger(\mathbf{k}),\mathbf{j}_N] = 0$$

The two contributions to the charge and current considered above cannot represent the total current, since the continuity equation

$$\dot{Q}(\mathbf{r},t) + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0 \tag{20.5}$$

cannot be satisfied inside the source. Thus, making use of field equations (15.7) and (15.8) with the interaction H', we find

$$\dot{Q}_\pi(\mathbf{r},t) + \nabla \cdot \mathbf{j}_\pi = ef[\tau_2(t)\phi_1 - \tau_1(t)\phi_2]\boldsymbol{\sigma}(t) \cdot \nabla\rho(r) \tag{20.6}$$

and

$$\dot{Q}_N(\mathbf{r},t) + \nabla \cdot \mathbf{j}_N = ef\rho(r)\boldsymbol{\sigma}(t) \cdot \int d^3r'\,[\tau_1(t)\phi_2(\mathbf{r}',t) - \tau_2(t)\phi_1(\mathbf{r}',t)]\nabla\rho(r') \tag{20.7}$$

In the limit of a point source this gives

$$\dot{Q}_\pi(\mathbf{r},t) + \dot{Q}_N(\mathbf{r},t) + \nabla \cdot \mathbf{j}_\pi + \nabla \cdot \mathbf{j}_N = -\nabla \cdot \mathbf{j}_I$$

with

$$\mathbf{j}_I = ef\delta^3(\mathbf{r})\boldsymbol{\sigma}[\tau_1(t)\phi_2(\mathbf{r},t) - \tau_2(t)\phi_1(\mathbf{r},t)] \tag{20.8}$$

Thus, (20.5) is satisfied if the total current includes the "interaction current"[1] $\mathbf{j}_I$. The physical reason why we need $\mathbf{j}_I$ is the following. When a π^+ is emitted by a proton, it changes into a neutron, and the charge $Q_N(\mathbf{r})$ suddenly disappears and is transferred from the source to the meson cloud. The continuity equation thus requires a current to carry the charge from the nucleon to the cloud. Since the density of P-wave mesons is zero at the origin, they cannot immediately take over the charge from the nucleon. In a scalar theory, where H' is not proportional to $\boldsymbol{\sigma} \cdot \nabla\phi$ but merely to ϕ (e.g., S-wave mesons interact), the continuity equation is satisfied without any $\mathbf{j}_I$.

For an extended source the continuity equation is still not satisfied inside the source, even if $\delta^3(\mathbf{r})$ in (20.8) is replaced by $\rho(\mathbf{r})$. We can introduce additional[2] currents to make the continuity equation valid inside the source, but these are not uniquely defined and appear to be unimportant. At this stage, we have to appeal to a relativistic local theory, which in the static limit gives the three kinds of currents we considered. With these, (20.5) is satisfied for an extended source when integrated over any region which contains the source[1]

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[1] This current can be deduced formally by introducing complex fields and by replacing $\nabla\phi$ by $\nabla\phi - ie\mathbf{A}\phi$ in H'. This generates an addition $\mathbf{A} \cdot \mathbf{j}_I$ to H'.

[2] See R. H. Capps and R. G. Sachs, *Phys. Rev.*, **96**:540 (1954); and R. H. Capps and W. G. Holladay, *Phys. Rev.*, **99**:931 (1955).

$$\int_V d^3r[\dot{Q}(\mathbf{r},t) + \nabla \cdot \mathbf{j}] = 0 \tag{20.9}$$

Summarizing this discussion, we can say that the predictions of the static model for electric phenomena may not be as strong as those for purely pionic problems.

Having established the form of the current we needed, we shall investigate the properties of the various terms. The expectation values of $\mathbf{j}_\pi$ and $Q_\pi(\mathbf{r})$ between physical nucleons, which involve the P-wave parts, were investigated in the last chapter. Since, however, the operators are bilinear in ϕ, they contain not only other angular momenta but also cross terms between S and P waves or between D and P waves, for example. These types of terms are particularly important for electric-dipole transitions, which require a change of parity and of angular momentum by one unit. Since we shall be concerned with photons of wavelengths $\lambda\!\!\!^- \leqslant 10^{-13}$ cm, or $k_{\text{photon}} \geqslant 200$ Mev, a multipole expansion may, however, not be suitable for this term. The current $\mathbf{j}_I$ involves principally S waves,[2] because it is zero outside the source, and inside it,

$$k_{\text{photon}} \times r_{\text{source}} \approx \frac{k_{\text{photon}}}{k_{\max}} \leqslant \frac{2}{3} \qquad \text{for } \lambda\!\!\!^-_{\text{photon}} \geqslant 5 \times 10^{-14}\,\text{cm}$$

It is only for higher photon energies that the other angular momenta begin to play a role for $\mathbf{j}_I$.

There is little we know about $\mathbf{j}_N$, since the naïve expectation that it is just the current of a Dirac particle for the bare proton and zero for the bare neutron gives a considerable departure from the observed magnetic moments. Fortunately, in the following, the expectation value of the total current between physical nucleon states will be important, and this can be related to the observed magnetic moments. Thus, let us consider a Fourier component $\mathbf{k}$ of the matrix elements of $\mathbf{j}_{\text{total}} \equiv \mathbf{j}$ between nucleon states. This must be a polar vector that can depend only on $\boldsymbol{\sigma}$, $\mathbf{k}$, and τ_3. The continuity equation which tells us that

$$\mathbf{k} \cdot \langle \xi \,|\, \mathbf{j}(\mathbf{k}) \,|\, \xi \rangle = 0$$

---

[1] This equation is not very strong and is also satisfied without $\mathbf{j}_I$.

[2] Because of this, it is also expected that $\mathbf{j}_I$ will not contribute appreciably to the magnetic moments.

therefore restricts the expectation value to[1]

$$\langle \xi \mid \mathbf{j}(-\mathbf{k}) \mid \xi \rangle \equiv \int \frac{d^3r}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} \langle \xi \mid j(\mathbf{r}) \mid \xi \rangle$$

$$= \langle \xi \mid i \frac{\boldsymbol{\sigma} \times \mathbf{k}}{(2\pi)^{3/2}} [\tau_3 F_v(k^2) + F_s(k^2)] \mid \xi \rangle \qquad (20.10a)$$

The static limit (e.g., $\mathbf{k} \to 0$) of the bracket term in (20.10*a*) corresponds to the total magnetic moments of the proton and neutron. This follows from the fact that

$$\mathcal{M} = \lim_{k \to 0} \frac{1}{2} \int [\mathbf{r} \times \mathbf{j}(\mathbf{r})] e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

$$= -\frac{i}{2} \lim_{\mathbf{k} \to 0} \nabla_{\mathbf{k}} \times \int \mathbf{j}(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

We thus note that

$$F_v(0) = \frac{\mathfrak{M}_p - \mathfrak{M}_n}{2} \qquad F_s(0) = \frac{\mathfrak{M}_p + \mathfrak{M}_n}{2} \qquad (20.10b)$$

Since the RBA term of $\mathbf{j}_\pi$ was shown in Chap. 19 to be $\propto e^{-2r}/r^2$ outside the source and since the other contributions arise from the source and should be even more concentrated around the origin, we expect the form factors F_v and F_s to be approximately equal to their static value for $k \lesssim 2$,[2] and we shall use these values. Since we know the expectation value of $\mathbf{j}_\pi$, we could use (20.10*a*) to determine $\mathbf{j}_N$ more specifically. However, our applications will always involve the expectation value of the total current, so that we need not do this.

20.2. The Production Amplitude. The first process that we shall turn to is the photoproduction of mesons, that is, reactions of the type $\gamma + N \to N + \pi$. This process involves not only the pion-nucleon system but also the radiation field. For its proper treatment we have to analyze the Hamiltonian

$$H = H_{\pi+N} + H_\gamma + H_{\text{int}} \qquad (20.11)$$

---

[1] We shall actually consider $\mathbf{j}(-\mathbf{k})$, since this is more useful for later discussions.

$$\mathbf{j}(+\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, \mathbf{j}(\mathbf{r})$$

[2] These form factors are the same as those in electron-proton scattering, and the analysis of these experiments bears this out. See R. Hofstadter, F. Bumiller, and M. R. Yearian, *Revs. Modern Phys.*, **30**:482 (1958).

where H_γ is the Hamiltonian of the free photons and $H_{\rm int}$ is the interaction of the photons with the charged particles.

We shall not go into the details of the quantization of the radiation field[1] but shall merely mention the pertinent facts. The free Hamiltonian H_γ describes the free photons which are massless particles with spin 1 but with spin components j_z only parallel or antiparallel to its momentum. In an angular-momentum expansion, we find that there are no one-particle states with zero angular momentum. For $l \neq 0$ they may have either parity. Photons with parity $(-1)^l$ are called electric l-pole photons, and those with parity $(-1)^{l+1}$ are called magnetic l-pole photons. The interaction Hamiltonian is

$$H_{\rm int} = -\int d^3r\, \mathbf{A}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) \tag{20.12}$$

where the vector potential $\mathbf{A}(\mathbf{r})$ is built up, in the usual manner of emission and absorption operators for photons.

Our goal is to find the transition probability from an initial state $\gamma + N$ to a final state $N + \pi$. We shall calculate it with the aid of the "golden rule" (8.27). This is simplified by the facts that $H_{\rm int}$ is weak and that in a perturbation[2] development in powers of e^2 we can replace T_{fi} by the corresponding matrix element of $H_{\rm int}$.

Accordingly, we shall work out

$$\begin{aligned}\langle \text{out}, N + \pi \,|\, H_{\rm int} \,|\, p + \gamma\rangle &= -\int \frac{d^3r}{(2\pi)^{3/2}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(2k)^{1/2}}\, \boldsymbol{\epsilon}\langle \text{out}, N + \pi \,|\, \mathbf{j}(\mathbf{r}) \,|\, N\rangle \\ &= \boldsymbol{\epsilon}\langle \text{out}, N + \pi \,|\, \mathbf{j}(-\mathbf{k}) \,|\, N\rangle (2k)^{-1/2} \end{aligned} \tag{20.13}$$

Here we have assumed an initial photon with momentum $\mathbf{k}$ and polarization vector $\boldsymbol{\epsilon}$, normalized to one particle per unit volume, so that[3]

$$\langle 0 \,|\, \mathbf{A}(\mathbf{r}) \,|\, \gamma\rangle = e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\boldsymbol{\epsilon}}{(2k)^{1/2}(2\pi)^{3/2}}$$

We shall first derive an integral equation for the matrix element of the

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[1] The quantization is similar to the treatment developed in this book, except that the spins of the photons are 1 and all photons are transverse. See, e.g., L. I. Schiff, "Quantum Mechanics," 2d ed., pp. 196–210, McGraw-Hill Book Company, Inc., New York, 1955.

[2] The pion interaction is treated "exactly."

[3] As in the case of the atomic photoeffect, the vector field $\mathbf{A}$ can also be considered a classical field. This effectively gives the same result as (20.13), in which we have already taken the matrix element with respect to the photon variables.

Fourier transform $\mathbf{j}(-\mathbf{k})$ of the total current. By means of (19.1) we can obtain a relation similar to the Low equation:[1]

$$
\begin{aligned}
\langle \text{out}, N + \pi_{\mathbf{k}'\alpha} \mid \mathbf{j}(-\mathbf{k}) \mid N \rangle &= \langle \xi' \mid [B_\alpha(\mathbf{k}'), \mathbf{j}(-\mathbf{k})] \mid \xi \rangle \\
&= \langle \xi' \mid [a_\alpha(\mathbf{k}') + \int_0^\infty V_\alpha(\mathbf{k}')\, e^{i\omega t}\, dt, \mathbf{j}(-\mathbf{k})] \mid \xi \rangle \\
&= \langle \xi' \mid [a_\alpha(\mathbf{k}'), \mathbf{j}(-\mathbf{k})] \mid \xi \rangle + \mathop{\mathrm{S}}_{n} \left[\frac{\langle \xi' \mid V_\alpha(\mathbf{k}') \mid \text{out}, n \rangle \langle \text{out}, n \mid \mathbf{j}(-\mathbf{k}) \mid \xi \rangle}{E_n - \omega - i\epsilon} \right. \\
&\quad \left. + \frac{\langle \xi' \mid \mathbf{j}(-\mathbf{k}) \mid \text{out}, n \rangle \langle \text{out}, n \mid V_\alpha(\mathbf{k}') \mid \xi \rangle}{E_n + \omega} \right] \\
&= \langle \xi' \mid [a_\alpha(\mathbf{k}'), \mathbf{j}(-\mathbf{k})] \mid \xi \rangle + \frac{1}{\omega} (\langle \xi' \mid \mathbf{j}(-\mathbf{k}) \mid \xi'' \rangle \langle \xi'' \mid V_\alpha(\mathbf{k}') \mid \xi \rangle \\
&\quad - \langle \xi' \mid V_\alpha(\mathbf{k}') \mid \xi'' \rangle \langle \xi'' \mid \mathbf{j}(-\mathbf{k}) \mid \xi \rangle) \\
&\quad + \mathop{\mathrm{S}}_{n}{}' \left[\frac{\langle \xi' \mid V_\alpha(\mathbf{k}') \mid \text{out}, n \rangle \langle \text{out}, n \mid \mathbf{j}(-\mathbf{k}) \mid \xi \rangle}{E_n - \omega - i\epsilon} \right. \\
&\quad \left. + \frac{\langle \xi' \mid \mathbf{j}(-\mathbf{k}) \mid \text{out}, n \rangle \langle \text{out}, n \mid V_\alpha(\mathbf{k}') \mid \xi \rangle}{E_n + \omega} \right] \qquad (20.14)
\end{aligned}
$$

The one-nucleon expectation value can readily be evaluated with the expression for $\mathbf{j}$, which we discussed earlier. With the help of

$$
[a_\alpha(\mathbf{k}), \phi_\beta] = \frac{(2\pi)^{-\frac{3}{2}} \delta_{\alpha\beta} e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\omega)^{\frac{1}{2}}}
$$

we find for the commutators

$$
\begin{aligned}
[a_\alpha(\mathbf{k}'), \mathbf{j}_N(-\mathbf{k})] &= 0 \\
[a_\alpha(\mathbf{k}'), \mathbf{j}_I(-\mathbf{k})] &= [a_\alpha(\mathbf{k}'), \int d^3r (\tau_1 \phi_2 - \tau_2 \phi_1) \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{\frac{3}{2}}} ef\rho(r)\boldsymbol{\sigma}] \\
&= \left(\frac{1}{2\pi}\right)^3 \frac{\rho(|\mathbf{k}' - \mathbf{k}|)}{(2\omega')^{\frac{1}{2}}} ef\boldsymbol{\sigma}(\tau_1 \delta_{\alpha,2} - \tau_2 \delta_{\alpha,1}) \\
[a_\alpha(\mathbf{k}'), \mathbf{j}_\pi(-\mathbf{k})] &= -\frac{e}{(2\pi)^{\frac{3}{2}}} \int d^3r\, [a_\alpha(\mathbf{k}'), \phi_1(\mathbf{r}) \nabla \phi_2(\mathbf{r}) - \phi_2(\mathbf{r}) \nabla \phi_1(\mathbf{r})]\, e^{i\mathbf{k}\cdot\mathbf{r}} \\
&= \frac{ie}{(2\pi)^3} \int \frac{d^3r}{(2\omega')^{\frac{1}{2}}} [\delta_{\alpha,2} \phi_1(\mathbf{r}) - \delta_{\alpha,1} \phi_2(\mathbf{r})](2\mathbf{k}' - \mathbf{k})\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}
\end{aligned}
\qquad (20.15)
$$

---

[1] We remind the reader that S' implies a sum over continuum states only.

The nucleon expectation value of the sum of the above commutators can be found by means of (19.3),

$$\langle \xi' \mid [a_\alpha(\mathbf{k}'), \mathbf{j}(-\mathbf{k})] \mid \xi \rangle = \frac{ef_r\rho(|\mathbf{k}'-\mathbf{k}|)}{(2\omega')^{\frac{1}{2}}(2\pi)^3}(\xi' \mid (\tau_1\delta_{\alpha,2} - \tau_2\delta_{\alpha,1}) \times \left[\boldsymbol{\sigma} - \frac{(2\mathbf{k}'-\mathbf{k})\boldsymbol{\sigma}\cdot(\mathbf{k}'-\mathbf{k})}{(\mathbf{k}'-\mathbf{k})^2+1}\right] \mid \xi) \tag{20.16}$$

and that of the current $\mathbf{j}$ is [see (20.10) with $F(k^2) \approx F(0)$]

$$\langle \xi' \mid \mathbf{j}(-\mathbf{k}) \mid \xi \rangle = i(\xi' \mid \boldsymbol{\sigma} \times \mathbf{k}\left(\frac{1+\tau_3}{2}\mathfrak{M}_p + \frac{1-\tau_3}{2}\mathfrak{M}_n\right) \mid \xi) \tag{20.17}$$

If we put the meson energy $\omega' = k$ (as is dictated by energy conservation), make use of (19.1), and remember that $\nabla \cdot \mathbf{A} = 0$ implies $\mathbf{k} \cdot \boldsymbol{\epsilon} = 0$, we find for the photoproduction amplitude

$$\begin{aligned}\langle \text{out}, \xi', \alpha\mathbf{k}' \mid H_{\text{int}} \mid \xi, \gamma_{\mathbf{k}} \rangle = -\boldsymbol{\epsilon}\Bigg\{&(\xi' \mid \frac{ef_r}{(2\pi)^3}\frac{\rho(|\mathbf{k}'-\mathbf{k}|)}{2\omega'} \\ &\times (\tau_1\delta_{\alpha,2} - \tau_2\delta_{\alpha,1})\left[\boldsymbol{\sigma} - \frac{2\mathbf{k}'\boldsymbol{\sigma}\cdot(\mathbf{k}'-\mathbf{k})}{(\mathbf{k}'-\mathbf{k})^2+1}\right] \mid \xi) + \frac{1}{2\omega'^2}f_r\frac{\rho(\mathbf{k}')}{(2\pi)^3} \\ &\times (\xi' \mid \left[i\boldsymbol{\sigma} \times \mathbf{k}\left(\frac{\mathfrak{M}_p + \mathfrak{M}_n}{2} + \tau_3\frac{\mathfrak{M}_p - \mathfrak{M}_n}{2}\right), i\boldsymbol{\sigma}\cdot\mathbf{k}'\tau_\alpha\right] \mid \xi) \\ &+ \mathop{\mathsf{S}'}_{n}\left[\frac{\langle \xi' \mid V_\alpha(\mathbf{k}') \mid \text{out}, n\rangle\langle \text{out}, n \mid \mathbf{j}(-\mathbf{k}) \mid \xi\rangle}{(E_n - \omega - i\epsilon)(2\omega')^{\frac{1}{2}}} \right. \\ &\left. + \frac{\langle \xi' \mid \mathbf{j}(-\mathbf{k}) \mid \text{out}, n\rangle\langle \text{out}, n \mid V_\alpha(\mathbf{k}') \mid \xi\rangle}{(E_n + \omega)(2\omega')^{\frac{1}{2}}}\right]\Bigg\}\end{aligned} \tag{20.18}$$

The one-nucleon terms are again equal to the renormalized Born approximation if the experimental values for the magnetic moments are used. The significance of these various contributions is the following. The first term on the right-hand side of (20.18) creates only charged mesons and corresponds to an absorption of the photon by the meson, as in the ordinary photoeffect. Of the two contributions, one arises from the current $\mathbf{j}_\pi$ (proportional to $\mathbf{k}'$), corresponding to an absorption by the meson in the cloud, and the other from the current $\mathbf{j}_I$ (proportional to $\boldsymbol{\sigma}$), corresponding to an absorption of the meson in the process of creation. These two contributions are shown diagrammatically in Fig. 20.1. The process of Fig. 20.1*b* mainly creates mesons in an *S* state, whereas that of Fig. 20.1*a* contains all angular momenta, owing to the retardation denominator $1/[(\mathbf{k}'-\mathbf{k})^2+1]$. This arises because the spatial extension of $\mathbf{j}_\pi$ (10^{-13} cm) is not small compared with the wavelength of the photon needed to create a real meson, so that a multipole expansion for this factor is not rapidly convergent. The terms with the magnetic moments correspond to an absorption of the meson by the magnetic moment of the nucleon accompanied by the emission of a meson, as illustrated in Fig. 20.2. They are the only terms which create neutral pions as well. One would expect these terms to be of the order $1/M$ compared with those of Fig. 20.1, but since $\mathfrak{M}_p - \mathfrak{M}_n = 4.7$, they may become comparable to the others.

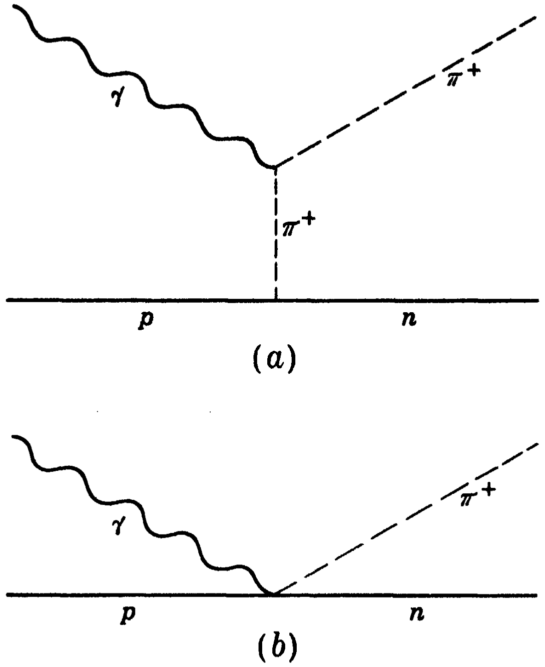

Fig. 20.1. Diagrams to represent the photoproduction of charged mesons caused by the terms in the matrix element which arise from (*a*) $[a_\alpha(\mathbf{k}'), \mathbf{j}_\pi(-\mathbf{k})]$ and (*b*) $[a_\alpha(\mathbf{k}'), \mathbf{j}_l(-\mathbf{k})]$.

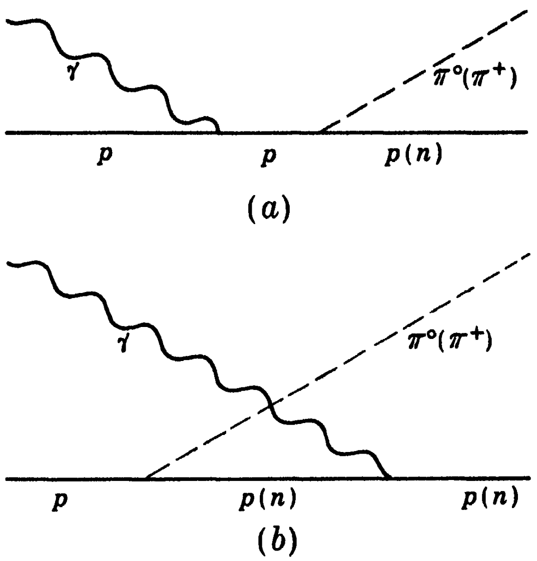

Fig. 20.2. Diagrams corresponding to the photoproduction of neutral and charged mesons due to the interaction of the photon with the magnetic moment of the nucleon.

The contributions contained in $\mathfrak{S}'_n$ correspond to a rescattering of the meson once it is created. Their evaluation is a complicated, but soluble, task.[1] Fortunately, the dominant rescattering correction can be obtained without solving the integral equation (20.18) for the production amplitude. From our experience with π-N scattering, we expect that the rescattering gives an enhancement of the amplitude in the 3-state and a suppression in the other states. Furthermore, it turns out that these corrections mainly affect the magnetic-moment term (Fig. 20.2) and not so much the photoelectric effects (Fig. 20.1). The reason is that in the process of Fig. 20.1*a* the photon is absorbed by a meson of the cloud, which is at some distance from the source and, therefore, has a reasonable chance of escaping without further interaction. Diagram 20.1*b* leads to mesons in *S* states which escape without further interaction.[1] On the other hand, the terms represented by diagram 20.2 are considerably affected by corrections of higher order in f. These can be calculated by introducing a current $\mathbf{j}'_N$, which depends only on the operators $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ (and not on the meson operators):

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[1] R. Omnès, *Nuovo cimento*, **8**:316 (1958).

$$\mathbf{j}'_N(-\mathbf{k}) = \mathbf{j}_v + \mathbf{j}_s$$

with

$$\mathbf{j}_v(-\mathbf{k}) \equiv \frac{i}{(2\pi)^{3/2}} \boldsymbol{\sigma} \times \mathbf{k} \tau_3 \frac{\mathfrak{M}_p - \mathfrak{M}_n}{2\mathfrak{r}_2}$$

$$\mathbf{j}_s(-\mathbf{k}) \equiv \frac{i}{(2\pi)^{3/2}} \boldsymbol{\sigma} \times \mathbf{k} \frac{\mathfrak{M}_p + \mathfrak{M}_n}{2\mathfrak{r}_1}$$

From the definition of this current we have

$$\langle \xi' \mid \mathbf{j}(-\mathbf{k}) \mid \xi \rangle = \langle \xi' \mid \mathbf{j}'_N(-\mathbf{k}) \mid \xi \rangle \qquad \text{and} \qquad [a_\alpha(\mathbf{k}'), \mathbf{j}'_N(-\mathbf{k})] = 0$$

Hence, on writing $\mathbf{j} = (\mathbf{j} - \mathbf{j}'_N) + \mathbf{j}'_N$, the first term gives the same commutator (20.15) but zero contribution to the magnetic-moment terms (20.17). That is to say, the amplitude for this part obeys the integral equation

$$\begin{aligned}\frac{1}{(2\omega')^{1/2}} \langle \text{out}, \xi', \alpha\mathbf{k}' \mid \boldsymbol{\epsilon} \cdot (\mathbf{j} - \mathbf{j}'_N) \mid \xi \rangle = -\boldsymbol{\epsilon} \Bigg\{ &(\xi' \mid \frac{ef_r}{(2\pi)^3} \frac{\rho(|\mathbf{k}' - \mathbf{k}|)}{2\omega'} \\ &\times (\tau_1 \delta_{\alpha,2} - \tau_2 \delta_{\alpha,1}) \left[\boldsymbol{\sigma} - \frac{2(\mathbf{k}' - \mathbf{k})\boldsymbol{\sigma} \cdot \mathbf{k}'}{(\mathbf{k}' - \mathbf{k})^2 + 1} \right] \mid \xi) \\ &+ \mathop{\mathsf{S}'}_n \Bigg[\frac{\langle \xi' \mid V_\alpha(\mathbf{k}') \mid \text{out}, n \rangle \langle \text{out}, n \mid \mathbf{j}(-\mathbf{k}) - \mathbf{j}'_N(-\mathbf{k}) \mid \xi \rangle}{(2\omega')^{1/2}(E_n - \omega - i\epsilon)} \\ &+ \frac{\langle \xi' \mid \mathbf{j}(-\mathbf{k}) - \mathbf{j}'_N(-\mathbf{k}) \mid \text{out}, n \rangle \langle \text{out}, n \mid V_\alpha(\mathbf{k}') \mid \xi \rangle}{(2\omega')^{1/2}(E_n + \omega)} \Bigg] \Bigg\}\end{aligned} \tag{20.19}$$

This is identical with (20.18), except that the magnetic-moment terms are missing. We argued that the rescattering corrections for the terms in (20.19) are insignificant, and to a fair approximation we keep here only the renormalized Born approximation. Turning to the matrix elements of $\mathbf{j}'_N$, we first observe that

$$\langle \text{out}, \xi', \mathbf{k}' \mid \mathbf{j}_v \mid \xi \rangle$$

---

[1] One might think that the *S*-wave scattering, which is not contained in our model, would significantly change this contribution and therefore the low-energy cross section. This does not occur, as was shown by S. D. Drell, M. H. Friedman, and F. Zachariasen, *Phys. Rev.*, **104**:236 (1956). The limiting form of the cross section obtained at low energies also holds in relativistic treatments and is known as the Kroll-Ruderman theorem. See N. M. Kroll and M. A. Ruderman, *Phys. Rev.*, **93**:233 (1954).

can be directly expressed in terms of scattering amplitudes, since it involves only matrix elements of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ between a nucleon and a "nucleon + pion" state. To this end, we have to expand in projection operators of eigenstates of angular momentum and isospin. Of these, only the matrix element for the transition into the $\frac{3}{2},\frac{3}{2}$-state becomes comparable with (20.19), whereas the other terms remain small. With the aid of (18.6), we can express the matrix element

$$\langle \text{out}, \xi', \mathbf{k}' \,|\, \tau_3 \boldsymbol{\sigma} \cdot \mathbf{k} \times \boldsymbol{\epsilon} \,|\, \xi \rangle$$

in terms of scattering amplitudes. This is most directly done by using a plane-wave representation. There we have to take the projection operators (19.5*a*) and change the density of state factors in (18.25) from $k(\pi\omega)^{-1}$ to $(4\pi k'\omega')^{-1}$:

$$\frac{f}{(16\pi^3\omega')^{\frac{1}{2}}} \langle \text{out}, \xi', \mathbf{k}'\alpha \,|\, \tau_3 \boldsymbol{\sigma} \cdot \mathbf{k} \times \boldsymbol{\epsilon} \,|\, \xi \rangle \simeq \frac{\sin \delta_3 e^{i\delta_3}}{16\pi^3 k'^3 \omega' \rho(|\mathbf{k} \times \boldsymbol{\epsilon}|)} \times (\delta_{3,\alpha} - \tfrac{1}{3}\tau_\alpha\tau_3)[2\mathbf{k}' \cdot \mathbf{k} \times \boldsymbol{\epsilon} - i\boldsymbol{\sigma} \cdot \mathbf{k}' \times (\mathbf{k} \times \boldsymbol{\epsilon})]$$

From this relation we immediately obtain[1]

$$\begin{aligned} \langle \text{out}, \xi', \mathbf{k}'\alpha \,|\, \boldsymbol{\epsilon} \cdot \mathbf{j}_v(-\mathbf{k}) \,|\, \xi \rangle \frac{1}{(2\omega')^{\frac{1}{2}}} &= \frac{\mathfrak{M}_p - \mathfrak{M}_n}{2\mathfrak{r}_2} \frac{1}{(2\omega')^{\frac{1}{2}}(2\pi)^{\frac{3}{2}}} \frac{1}{\rho(|\mathbf{k} \times \boldsymbol{\epsilon}|)} \langle \text{out}, \xi', \mathbf{k}'\alpha \,|\, \tau_3 \boldsymbol{\sigma} \cdot \mathbf{k} \times \boldsymbol{\epsilon} \,|\, \xi \rangle \\ &= f_r \frac{\mathfrak{M}_p - \mathfrak{M}_n}{2} \frac{e^{i\delta_3} \sin \delta_3}{f_r^2 k'^3} \frac{1}{16\omega'\pi^3 \rho(|\mathbf{k} \times \boldsymbol{\epsilon}|)} (\delta_{3,\alpha} - \tfrac{1}{3}\tau_\alpha\tau_3) \\ &\quad \times [2\mathbf{k}' \cdot \mathbf{k} \times \boldsymbol{\epsilon} - i\boldsymbol{\sigma} \cdot \mathbf{k}' \times (\mathbf{k} \times \boldsymbol{\epsilon})] \qquad (20.19a) \end{aligned}$$

This is just the RBA term of (20.18) multiplied by the ratio of the actual scattering amplitude to the RBA scattering amplitude. Regarding $\mathbf{j}_s$, we find that its matrix elements are insignificant compared with the leading terms that we have evaluated so far, namely, (20.19) and (20.19*a*). This is partially due to the fact that $\mathbf{j}_s$ cannot lead into the $\frac{3}{2},\frac{3}{2}$-state, and because $\mathfrak{M}_p + \mathfrak{M}_n < \frac{1}{4}(\mathfrak{M}_p - \mathfrak{M}_n)$. Thus, $\mathbf{j}_s$ will be dropped in the following. Since (20.19*a*) is then the only term[2] which produces π^0's, we can relate the π^0 cross section directly to pion scattering:

$$\begin{aligned} \sigma(\gamma_{\mathbf{k}} + N \rightarrow \pi^0_{\mathbf{k}'} + N) &= \left(\frac{\mathfrak{M}_p - \mathfrak{M}_n}{2}\right)^2 \frac{k}{f_r^2 k'} \frac{1}{\rho^2(|\mathbf{k} \times \boldsymbol{\epsilon}|)} \sigma(\pi^0_{\mathbf{k}\times\boldsymbol{\epsilon}} + N \rightarrow \pi^0_{\mathbf{k}'} + N) \qquad (20.19b) \end{aligned}$$

---

[1] Remember that the kinematics are now such that $\omega = k = \omega' = (k'^2 + 1)^{\frac{1}{2}}$ and δ_3 is to be taken at that energy.

[2] Actually there is also a small production amplitude of π^0's in S states due to the S scattering, which is not contained in our model.

Altogether, we get the following result as a fair approximation to the rather complicated production amplitude:

$$
\begin{aligned}
&\langle \text{out}, \pi_{\mathbf{k}'\alpha} + N \,|\, H_{\text{int}} \,|\, N + \gamma_{\mathbf{k}} \rangle \\
&\quad = \frac{ef_r}{2\omega'} (\tau_2 \delta_{\alpha,1} - \tau_1 \delta_{\alpha,2}) \rho(|\mathbf{k}' - \mathbf{k}|) \left[\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} - 2\mathbf{k}' \cdot \boldsymbol{\epsilon} \frac{\boldsymbol{\sigma} \cdot (\mathbf{k}' - \mathbf{k})}{(\mathbf{k}' - \mathbf{k})^2 + 1} \right] \\
&\qquad \times (2\pi)^{-3} + f_r \frac{\mathfrak{M}_p - \mathfrak{M}_n}{2} \frac{e^{i\delta_3} \sin \delta_3}{\rho(|\mathbf{k} \times \boldsymbol{\epsilon}|) f_r^2 k'^3} (\delta_{3,\alpha} - \tfrac{1}{3}\tau_\alpha \tau_3) \\
&\qquad \times [2\mathbf{k}' \cdot \mathbf{k} \times \boldsymbol{\epsilon} - i\boldsymbol{\sigma} \cdot \mathbf{k}' \times (\mathbf{k} \times \boldsymbol{\epsilon})](16\pi^3 \omega)^{-1} \qquad (20.20)
\end{aligned}
$$

20.3. General Features of the Cross Section. With the aid of the "golden rule," the cross section can be worked out from the amplitude (20.20). The result is fairly complicated, but the physical significance and the important features of the various contributions can be understood by simple probability arguments. For the meson-cloud-creation process (Fig. 20.1*a*) the cross section is expected to be of the order of

σ = (probability that the photon gets into the meson cloud) × (probability that a meson is present) × (probability for absorption of the photon by a meson in a *P* state) × (probability that the meson leaves the cloud per unit time) × (incident photon flux)$^{-1}$

$$
\begin{aligned}
&= \left(\frac{1}{\omega^3 L^3}\right) \left(\frac{f_r^2}{4\pi} k'^2\right) \left(\frac{e^2}{4\pi} k^2\right) (k')(L^3) \\
&= \frac{f_r^2}{4\pi} \frac{e^2}{4\pi} \frac{k'^3 k^2}{\omega^3} \qquad (20.21a)
\end{aligned}
$$

This cross section shows a strong increase with meson momentum and is vanishingly small at the production threshold. The contribution from $\mathbf{j}_I$ (Fig. 20.1*b*) is similar but lacks the factors k'^2 and k^2, since the meson is produced in an *S* state, so that[1]

$$
\sigma_I = \frac{f_r^2}{4\pi} \frac{e^2}{4\pi} \frac{k'}{\omega^3} \qquad (20.21b)
$$

The energy dependence of these two terms is illustrated in Fig. 20.3. These estimates again reflect the features of the exact expressions only at low energies, where the probabilities are small. In general, the threshold behavior is dictated by parity and angular-momentum conservation. The various possibilities are summarized in Table 20.1. From this table we learn that the linear increase of σ_I with momentum

---

[1] More accurately, one obtains $\sigma_I \propto k'/\omega$, which is the energy dependence plotted in Fig. 20.3. Close to the threshold for meson production $\omega \approx 1$, and the difference between the two expressions is small.

Table 20.1

Type of pion	$\gamma + p \rightarrow \pi^+ + n$	Momentum dependence of cross section
Scalar pion	Electric dipole $(\mathscr{P} = -1)$ $l = 1, J = \frac{1}{2}, \frac{3}{2}$	k'^3
	Magnetic dipole $(\mathscr{P} = +1)$ $l = 0, J = \frac{1}{2}$ $(l = 2, J = \frac{3}{2}, \frac{5}{2})$	k' (k'^5)
Pseudoscalar pion	Electric dipole $(\mathscr{P} = -1)$ $l = 0, J = \frac{1}{2}$ $(l = 2, J = \frac{3}{2}, \frac{5}{2})$	k' $(k')^5$
	Magnetic dipole $(\mathscr{P} = +1)$ $l = 1, J = \frac{1}{2}, \frac{3}{2}$	k'^3

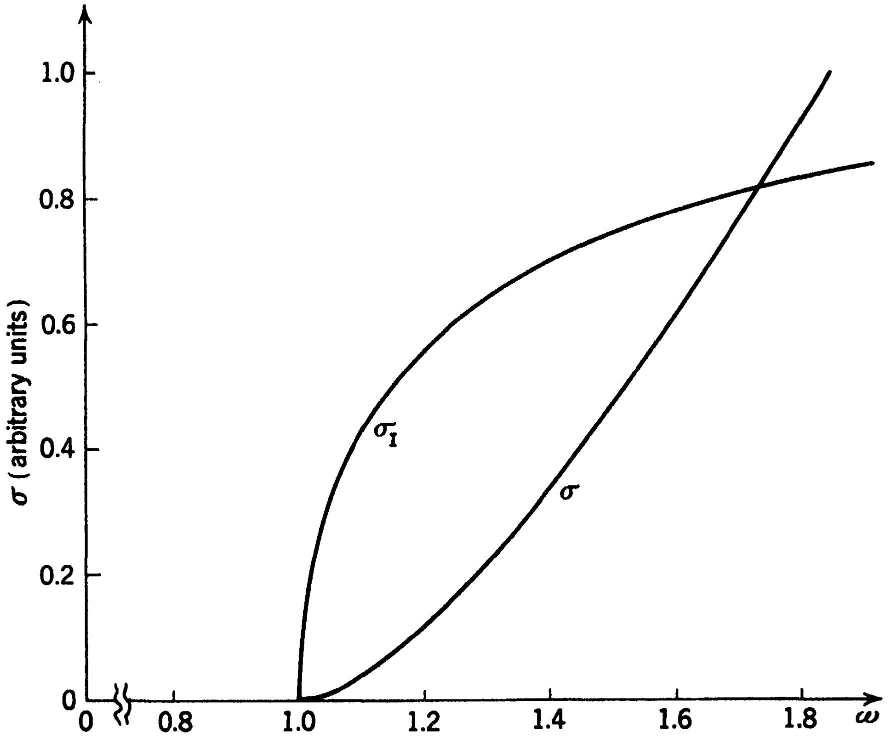

Fig. 20.3. Qualitative energy dependence of the photoproduction cross section from the meson cloud, σ (see Fig. 20.1*a*), and from the interaction current, σ_I, (see Fig. 20.1*b*).

is a manifestation of the pseudoscalar nature of the pion. For scalar particles, the dominant low-energy process would be the absorption of an electric-dipole photon by the meson in an *S* state which is then emitted in a *P* state. This has a threshold behavior not $\propto k'$ but $\propto k'^3$, like the atomic photoeffect. This difference is also reflected in the angular distribution, which is isotropic for pions emitted in *S* waves and

$\propto \sin^2\theta$ for P waves, since the angular momentum is in the direction of the photon momentum.

Near the resonant energy ($\omega' \sim 2$, or 300-Mev γ rays in the laboratory system), the contribution from the isovector current $\mathbf{j}_V$ becomes important. Whenever one state dominates the production process, the angular and energy dependence of the cross section is quite simple. Thus, the angular distribution at the $\frac{3}{2}$, $\frac{3}{2}$-resonance is obtained by averaging over the two possible spin states of the initial nucleon. The nucleon spin can be quantized along the photon spin direction, which can be taken to be parallel to the photon momentum. Further averaging over the photon spin is unnecessary, and we obtain [see (16.37)]:

Angular momentum	Initial state	Final state	Relative contribution to $d\sigma/d\Omega$
$J = \frac{3}{2}, J_z = \frac{3}{2}$	$\lvert\uparrow\gamma, \uparrow N\rangle$	$\lvert\uparrow 1\rangle$	$\lvert Y_1^1\rvert^2$
$J = \frac{3}{2}, J_z = \frac{1}{2}$	$\lvert\uparrow\gamma, \downarrow N\rangle$	$(\frac{1}{3})^{\frac{1}{2}}[(\frac{1}{3})^{\frac{1}{2}}\lvert\downarrow 1\rangle + (\frac{2}{3})^{\frac{1}{2}}\lvert\uparrow 0\rangle]$	$\frac{1}{9}(\lvert Y_1^1\rvert^2 + 2\lvert Y_1^0\rvert^2)$

The factor of $(\frac{1}{3})^{\frac{1}{2}}$ for the antiparallel-spin case arises because the initial state has a probability of $\frac{1}{3}$ for being in the $J = \frac{3}{2}$ state and of $\frac{2}{3}$ for being in the $J = \frac{1}{2}$ state. The total angular distribution for mesons produced in the $J = \frac{3}{2}$ state is thus

$$\frac{d\sigma}{d\Omega} \propto \frac{10}{9}\,|Y_1^1|^2 + \frac{2}{9}\,|Y_1^0|^2 \propto 2 + 3\sin^2\vartheta \tag{20.22}$$

Since the $T = \frac{3}{2}$, $T_z = -\frac{1}{2}$ state contains twice as many neutral pions as charged ones [see (16.35)], we obtain 2 for the ratio of neutral- to charged-meson production in the $T = J = \frac{3}{2}$ state. Near the resonant energy, the energy dependence of the cross section from (20.19*a*) is given by

$$\sigma = \frac{\pi}{k'^2}\frac{\Gamma\Gamma_\gamma}{(\omega' - \omega_r)^2 + (\Gamma/2)^2} \qquad \text{with } \Gamma_\gamma = \frac{\Gamma}{f_r^2\rho^2(k)}\left(\frac{\mathfrak{M}_p - \mathfrak{M}_n}{2}\right)^2\frac{\omega'}{k'} \tag{20.23}$$

20.4. Comparison with Experiment. The experimental total cross section for the photoproduction of charged and neutral mesons is shown in Fig. 20.4. Both the resonance and the threshold behavior of the cross section behave qualitatively as predicted by the simple static model. For neutral mesons the experimental cross section is much less than the charged one at low energies, but it catches up close to the resonance energy. Over this whole range of energies, the π^0 cross section is well described by (20.19*b*) and (20.23). The solid curve of Fig. 20.4 is just the contribution from $\mathbf{j}_v$ as given by (20.19*a*). The

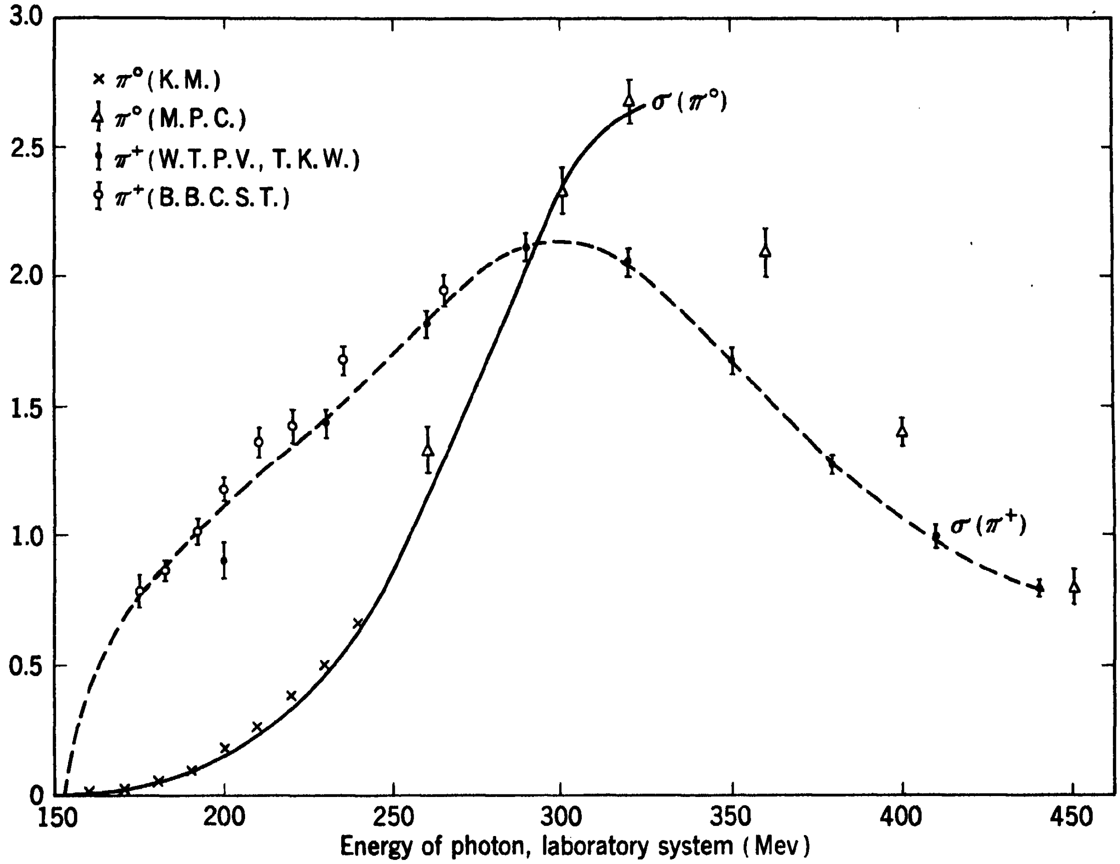

Fig. 20.4. Total cross section for π^+ and π^0 photoproduction on protons, as a function of photon laboratory energy. The π^0 experimental data were obtained by L. S. Koester and F. E. Mills, *Phys. Rev.*, **105**:1900 (1957); and by W. S. McDonald, V. Z. Peterson, and D. R. Corson, *Phys. Rev.*, **107**:577 (1957). The solid curve for the π^0 photoproduction corresponds to the contribution (20.19*a*) due to the current j_v alone (see Koester and Mills). The high-energy π^+ data are an average of those found by R. L. Walker, J. G. Teasdale, V. Z. Peterson, and J. I. Vette, *Phys. Rev.*, **99**:210 (1955); and by A. V. Tollestrup, J. C. Keck, and R. M. Worlock, *Phys. Rev.*, **99**:220 (1955). The lower-energy data were obtained by M. Beneventano, G. Bernardini, D. Carlson-Lee, G. Stoppini, and L. Tau, *Nuovo cimento*, **4**:323 (1956). The dashed curve is an arbitrary one to fit the experimental data and to illustrate the low-energy behavior (see Fig. 20.3).

excitation of π^+ mesons needs some refinements,[1] mainly due to recoil effects, but is quite well described by the theory. The additional contribution of the electric-dipole terms and of higher-order multipoles causes the total π^+ cross section to be larger than one-half that of the

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[1] See M. J. Moravcsik, *Phys. Rev.*, **104**:1451 (1956); G. F. Chew, *Phys. Rev.*, **106**:1337 (1957).

π^0, as would have been obtained according to the argument given at the end of the last section. The low-energy data fit (20.21) with $f_r^2/4\pi \approx 0.07$, in good agreement with the coupling constant found from meson-nucleon scattering.

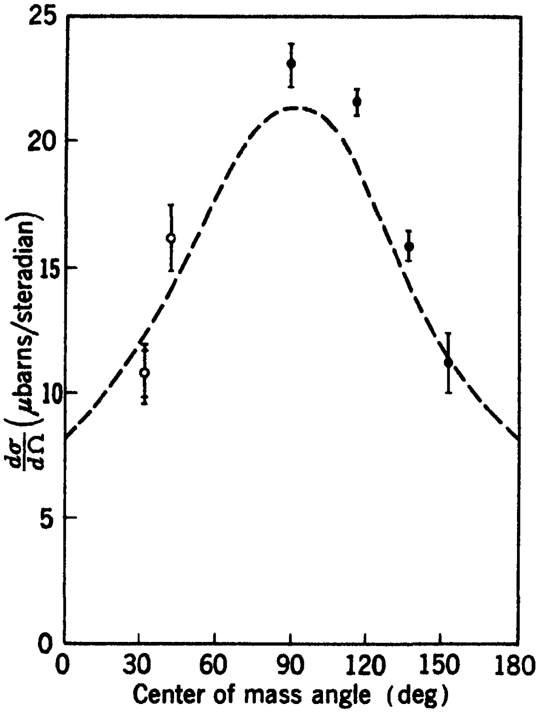

Fig. 20.5. Differential cross section for the production of π^0 mesons at a laboratory energy of 300 Mev. The experimental points stem from W. S. McDonald, V. Z. Peterson, and D. R. Corson, *Phys. Rev.*, **107**:577 (1957); D. C. Oakley and R. L. Walker, *Phys. Rev.*, **97**:1283 (1955); and Y. Goldschmidt-Clermont, L. S. Osborne, and M. Scott, *Phys. Rev.*, **97**:188 (1955). Dashed curve represents $2 + 3 \sin^2 \vartheta$, as predicted from a pure $J = T = \frac{3}{2}$ contribution to the production process.

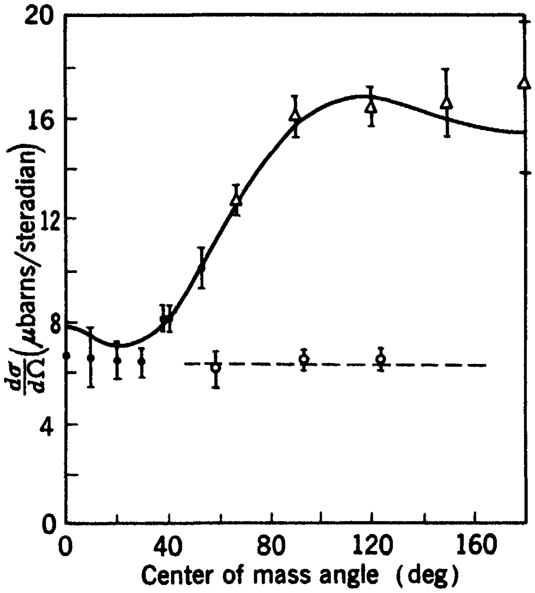

Fig. 20.6. Differential cross section for the production of π^+ mesons at 175 Mev (open circles) and at 260 Mev. The former data are due to Beneventano et al. (see legend for Fig. 20.4). The sources for the higher-energy data are also cited in the legend for Fig. 20.4. The dashed curve is a straight line, corresponding to pure S-state production. The higher-energy curve is taken from "Proceedings of the 7th Annual Rochester Conference," p. II-58, Interscience Publishers, Inc., New York, 1957.

The angular distribution of π^0 mesons in the resonance region also agrees with our model and is given fairly well by (20.22). The comparison is made at 300 Mev in Fig. 20.5. For π^+ mesons, the model predicts that, near threshold, only S-state mesons are produced, with a consequent spherically symmetric angular distribution. This is seen to be so within experimental error at 175 Mev (25 Mev above threshold) in Fig. 20.6. At higher energies, there are interfering contributions from the electric and magnetic absorptions. Since the latter has a phase $e^{i\delta_3}$, the interference term changes rapidly near the resonance. This is reflected in a shift of the maximum of the angular distribution from backward to forward directions in this region. A comparison between the angular distribution and the predictions of the static model are shown below the resonance energy, at 260 Mev, in Fig. 20.6. Thus, the theory successfully predicts subtle effects, such as the relative phase of the electric and magnetic amplitudes.

Considering the uncertainty of the static model for the description of electromagnetic phenomena, as well as the neglect of S-state nucleon-meson interactions, the agreement above is remarkable.[1] In particular, it must be stressed that the photoproduction of charged and neutral pions provides two new independent measurements of f_r and that both agree with the value deduced from pion scattering.

20.5. Compton Scattering. Another electromagnetic phenomenon for which the $\frac{3}{2},\frac{3}{2}$-state is important is the scattering of photons by nucleons. This is an effect of second order in e, and we shall mention only the pertinent points. The normal (Thomson) scattering involves the translational degrees of freedom in the same way as the Compton effect on the electron. The incident photon shakes the proton, which then emits the scattered photon. Because of the larger mass of the proton, this gives an exceedingly small cross section $\sim e^2/M^2 \sim 10^{-31}$ cm^2. However, because of the excited state of the nucleon, there is the possibility of a resonance scattering in which the incident photon excites the $J = T = \frac{3}{2}$ state, which subsequently decays by emission of the scattered photon.[2] In the classical picture this means that the photon acts on the magnetic moment of the nucleon, thereby creating a forced gyration of the nucleon spin. This will give a resonance scattering of the usual form[3] [compare (20.23)]

$$\sigma_{\text{resonance}} = \pi \lambda\!\!\!^{-2} \frac{\Gamma_\gamma^2}{(E_\gamma - E)^2 + (\Gamma/2)^2}$$

where $\lambda\!\!\!^{-}$ is the wavelength of the incident photon, or $1/E_\gamma$; E_γ = resonance energy $\sim$ 300 Mev in the laboratory system; and the other symbols are as defined in (20.23). The cross section is much less than the geometrical limit $\pi\lambda\!\!\!^{-2}$, since the excited state will preferentially decay by the emission of a pion ($\Gamma_\lambda \ll \Gamma$), but it is much larger than the Thomson scattering (e^2/M^2). Modern experimental techniques make it possible to verify these predictions of the theory in spite of the small cross sections involved.

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[1] In the comparison of the static model with experiment, the energies which appear in the formulas are always assumed to refer to the center-of-mass system.

[2] For an analysis based on the static model, see, e.g., W. J. Karzas, W. K. R. Watson, and F. Zachariasen, *Phys. Rev.*, **110**:253 (1958).

[3] See J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," p. 394, John Wiley & Sons, Inc., New York, 1952.

Further Reading

G. F. Chew and F. E. Low, *Phys. Rev.*, **101**:1579 (1956).
G. F. Chew, Theory of Pion Scattering and Photoproduction, in "Handbuch der Physik," Springer-Verlag, Berlin (to be published).

CHAPTER 21

Nuclear Forces

21.1. Introduction: Classical Calculation of Nuclear Interaction Energy. In this last chapter we shall use the static model to calculate the long-range, or external, region of the forces between two nucleons due to the "exchange" of mesons. Although historically the Yukawa potential was the beginning of meson theory, this problem is not the most clear-cut test of the static model. A static potential should be able to account for all the low-energy properties (e.g., deuteron binding energy quadrupole moment, and nucleon-nucleon-scattering phase shifts) of the two-nucleon system. On the basis of range arguments alone, the external region of the potential should be due to the exchange of a single pion and should therefore be given fairly accurately by the static model. The exchange of two mesons should be partially responsible for the shorter-range behavior but will be much more strongly influenced by recoil effects (neglected by our model) and by the nucleon source structure (e.g., $k_{\max}$). Furthermore, other than P-wave mesons will contribute to it, e.g., mesons in S states from a ϕ^2 term. As we discussed in an earlier chapter, the exchange of K mesons will give rise to a still shorter range force, and these effects are presumably hidden in the source.

We can calculate the classical interaction energy between two sources a distance $\mathbf{r}_0$ apart in a manner analogous to that of Chap. 9. Because the interaction energy is proportional to ρ^2 rather than to ρ, we obtain a cross term in the energy of two sources:

$$\rho(\mathbf{r},\mathbf{r}_0) = \rho(\mathbf{r}) + \rho(\mathbf{r} - \mathbf{r}_0) \tag{21.1}$$

The classical field solution was obtained in Chap. 16 [see (16.4)] for a neutral pseudoscalar field in the presence of a single source. It is straightforward to generalize this to a symmetrically coupled field to obtain, for static $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$,

$$\phi_\alpha(\mathbf{r}) = f\tau_\alpha \boldsymbol{\sigma}\cdot\nabla Y_0(\mathbf{r}) \tag{21.2a}$$

with

$$Y_0(\mathbf{r}) = \int d^3r'\,\rho(\mathbf{r}')\,\frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \tag{21.2b}$$

In the presence of two sources, the interaction Hamiltonian H' becomes

$$H' = f\sum_\alpha \int [\rho(\mathbf{r}) + \rho(\mathbf{r}-\mathbf{r}_0)]\boldsymbol{\sigma}\cdot\nabla\phi_\alpha\tau_\alpha\,d^3r \tag{21.3}$$

By substituting (21.2) into (21.3), we find that the interaction energy generated by one source at the position of the other one is (a and b refer to the two sources)

$$\mathscr{E}_0(2) = \mathscr{V} = f^2\sum_{\alpha=1}^{3}\tau_{a\alpha}\tau_{b\alpha}\int \rho(\mathbf{r}')\rho(\mathbf{r}'-\mathbf{r}_0)(\boldsymbol{\sigma}_a\cdot\nabla_r)(\boldsymbol{\sigma}_b\cdot\nabla_r)\frac{e^{-|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}\,d^3r\,d^3r' \tag{21.4}$$

For point sources, $\rho(\mathbf{r}) = \delta^3(\mathbf{r})$, this gives

$$\mathscr{V}(\mathbf{r}_0) = \frac{f^2}{4\pi}\sum_{\alpha=1}^{3}\tau_{a\alpha}\tau_{b\alpha}\frac{1}{3}\left[\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b + S_T\left(\frac{3}{r_0^2}+\frac{3}{r_0}+1\right)\right]\frac{e^{-r_0}}{r_0} - f^2\sum_{\alpha=1}^{3}\tau_{a\alpha}\tau_{b\alpha}\delta^3(\mathbf{r}_0)\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b \tag{21.5}$$

where S_T is the interaction energy between two dipoles (tensor force):

$$S_T = \frac{3\boldsymbol{\sigma}_a\cdot\mathbf{r}_0\boldsymbol{\sigma}_b\cdot\mathbf{r}_0}{r_0^2} - \boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b$$

With finite sources, the long-distance behavior of the interaction energy is the same, but the δ-function potential is spread out over the source and gives for the last term of (21.5)

$$-f^2\sum_{\alpha=1}^{3}\tau_{a\alpha}\tau_{b\alpha}\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b\int \rho(\mathbf{r}')\rho(\mathbf{r}'-\mathbf{r}_0)\,d^3r' \tag{21.6}$$

In a perturbation-theory calculation of the two-body force, the lowest-order contribution arises from the diagram shown in Fig. 21.1 due to the exchange of a single meson between bare nucleons. The resulting potential is

$$\mathscr{V}(\mathbf{r}_0) = -\mathop{\mathbf{S}}_{n}\frac{(0\,|H'|\,n)(n\,|H'|\,0)}{E_0 - E_n} = -f^2\int \boldsymbol{\sigma}_a\cdot\mathbf{k}\,\boldsymbol{\sigma}_b\cdot\mathbf{k} \times \frac{\rho^2(k)}{\omega^2}\sum_{\alpha=1}^{3}\tau_{a\alpha}\tau_{b\alpha}\,e^{i\mathbf{k}\cdot\mathbf{r}_0}\,\frac{d^3k}{(2\pi)^3} \tag{21.7}$$

and corresponds exactly to that found in the classical limit. On the basis of our discussion in earlier chapters, we should expect the dominant contribution of the static model to be given by the same potential with f replaced by f_r. We shall see that this is actually the case.

The charge dependence of the potential (21.7) originates from $\boldsymbol{\tau}_a \cdot \boldsymbol{\tau}_b$, which has the opposite sign, depending on whether the isospins of the nucleons are parallel or antiparallel. Its expectation value is 1 in the isotriplet state and -3 in the isosinglet state. That the force depends on the relative orientations of the isospins was to be expected, since the possibilities of meson exchange are different in the two states. For instance, two protons can exchange only π^0's, whereas in the isosinglet state, nucleons can exchange π^0's and charged mesons. Correspondingly, the force is stronger in the isosinglet state. The opposite sign is connected with our earlier remark concerning the sign of the π^0 coupling constant to the proton and neutron.

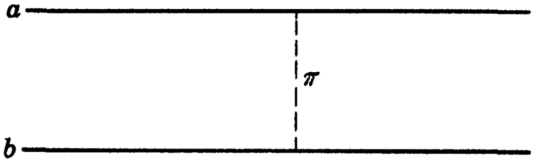
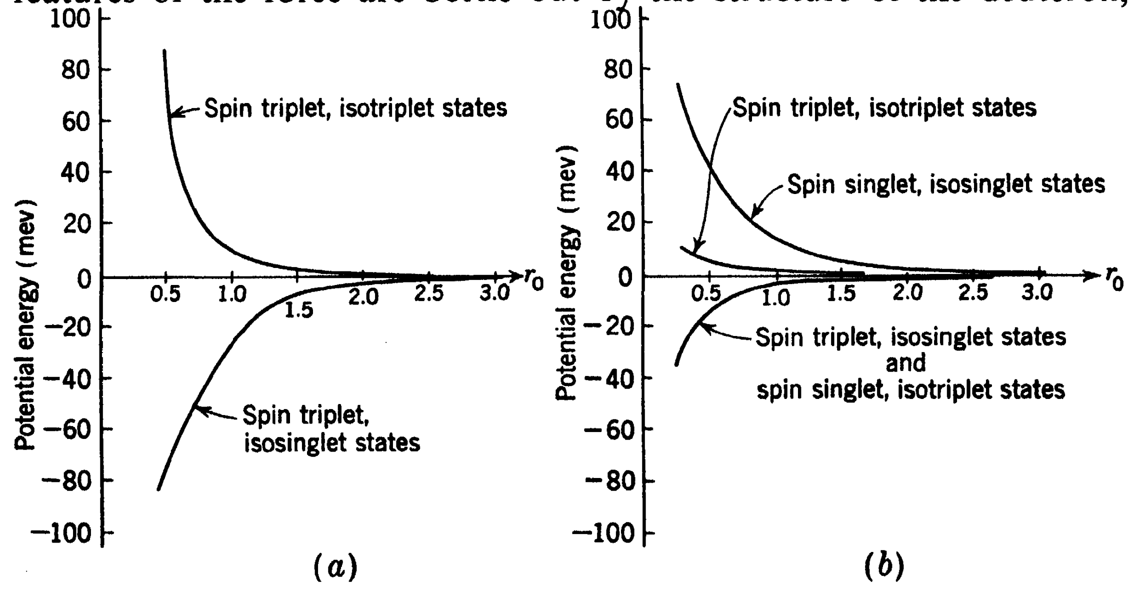


Fig. 21.1. Diagram for the one-meson-exchange contribution to nuclear forces.

The spin dependence, which comes predominantly from a factor $(\boldsymbol{\sigma}_a \cdot \mathbf{r}_0\, \boldsymbol{\sigma}_b \cdot \mathbf{r}_0)/r_0^2$, reflects the overlap of the meson clouds, each of which is proportional to $(\boldsymbol{\sigma} \cdot \mathbf{r}_0)$. For typical relative orientations of classical spins and of the distance, the factor $\boldsymbol{\sigma}_a \cdot \mathbf{r}_0\, \boldsymbol{\sigma}_b \cdot \mathbf{r}_0/r_0^2$ has the values illustrated in Fig. 21.3. For the various quantum states, the behavior of the force is illustrated in Fig. 21.2 and Table 21.1. These features of the force are borne out by the structure of the deuteron,



(*a*) (*b*)

Fig. 21.2. Plot of the longest-range contribution to the nuclear potential with $f_r^2/4\pi = 0.08$. (*a*) The tensor contribution to the potential energy. (*b*) Plot of the central force.

Table 21.1

CHARACTER OF ONE-MESON-EXCHANGE FORCE IN VARIOUS STATES

Spin	Isospin	Tensor	Central	Short-range
Triplet	Singlet	Strong attraction	Moderate attraction	Moderate repulsion
Triplet	Triplet	Moderate repulsion	Weak repulsion	Weak attraction
Singlet	Triplet		Moderate attraction	Moderate repulsion
Singlet	Singlet		Strong repulsion	Strong attraction

which is an isosinglet, triplet-spin state and is elongated in its spin direction (positive quadrupole moment). Such a state corresponds to the deepest value that the second-order potential can attain. For other states this potential can also be attractive, but less strongly so. This is in accord with the empirical fact that the total *n*-*n* force is also attractive but is not strong enough to give a bound state.

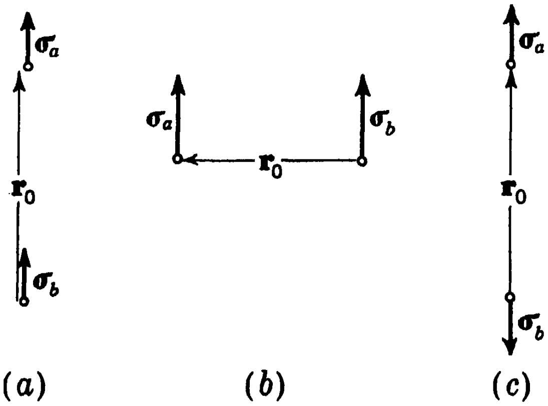


Fig. 21.3. Illustration of the value of $S'_T = \sigma_a \cdot \mathbf{r}_0 \sigma_b \cdot \mathbf{r}_0/\mathbf{r}_0^2$ for various nucleonic space and spin configurations. (*a*) $S'_T = 1$. (*b*) $S'_T = 0$. (*c*) $S'_T = -1$.

21.2. Static Potential, Quantum-mechanical. Before we proceed with the actual derivation of the potential, several remarks are in order. The derivation and use of the potential in the static-source limit are one in the spirit of the Born-Oppenheimer (molecular) approximation.[1] That is, the internucleon potential is calculated for nucleons a fixed distance $\mathbf{r}_0$ apart, where $\mathbf{r}_0$ is considered a parameter, and this potential is then used in a Schrödinger equation to calculate the properties of the nucleon-nucleon system. It is clear that it is more difficult to justify a Born-Oppenheimer treatment here than in the molecular problem, where the expansion is in terms of $(m_e/M)^{\frac{1}{4}} \sim \frac{1}{6}$, rather than $(\mu/M)^{\frac{1}{4}} \sim \frac{2}{3}$, where M is the nucleon mass. Nevertheless, it is only in the static limit that a potential can be defined at all consistently,[1] and it was also in this way that the force was defined for the neutral scalar theory in Chap. 10. It is clear that the potential derived in the above manner neglects corrections of order μ/M as well as those due to nuclear momenta $\mathbf{p}$ of order p/M.

---

[1] M. Born and J. R. Oppenheimer, *Ann. Physik*, **84**:457 (1927). See also L. I. Schiff, "Quantum Mechanics," 2d ed., p. 299, McGraw-Hill Book Company, Inc., New York, 1955.

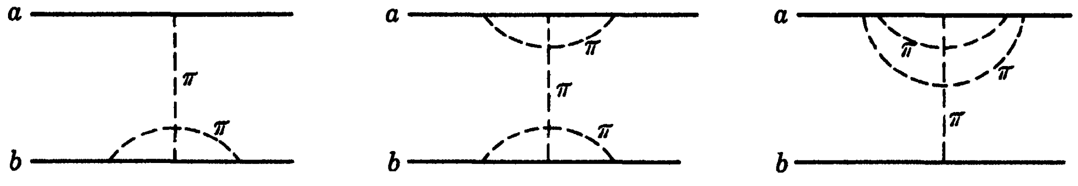


Fig. 21.4. Typical contributions to nuclear forces that are responsible for changing f into f_r.

The corrections to (21.7) of higher order in f^2 are substantial and difficult to calculate. Fortunately, for large distances, diagrams of the form shown in Fig. 21.4 are dominant, and their effect is simply to change f into f_r. The two-meson-exchange diagrams of Fig. 21.5 have range $\frac{1}{2}$, and their contribution can be related to meson-nucleon scattering. We shall only mention how they can be computed, since they are strongly influenced by meson-meson interactions and nonadiabatic corrections.[2]

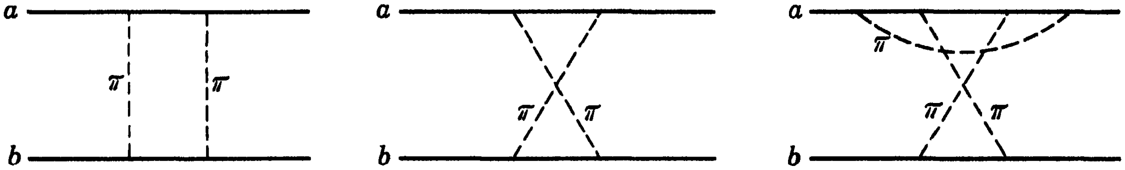


Fig. 21.5. Typical two-meson-exchange diagrams for nuclear forces.

Our results will be derived in the spirit of a Heitler-London approximation. Thus, for calculating the energy of the two-nucleon system, we shall, in first approximation, use a state wherein both nucleons have an undistorted meson cloud. This state can be written, with (17.6), as[3]

$$|\,\xi_a, \xi_b, \mathbf{r}_0\rangle = R_a R_b \,|\, \xi_a, \xi_b, \mathbf{r}_0) \tag{21.8}$$

---

[1] A potential is defined to be an energy-independent quantity that follows from field theory. For various definitions within this context, see K. Nishijima, *Suppl. Progr. Theoret. Phys.* (*Kyoto*), **3**:138 (1956). Whether the above "consistent" potential is reliable is quite a different question. For a discussion of this point, see J. M. Charap and S. Fubini, *Nuovo cimento*, **14**:540 (1959) and **15**:73 (1960).

[2] See S. Machida and T. Toyoda, *Suppl. Progr. Theoret. Phys.* (*Kyoto*), **3**:106 (1956).

[3] For the following it must be remembered that the "dressing operator" R contains only meson-creation and $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ operators. Since the operators $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ belonging to different nucleons commute, we have $[R_a, R_b] = 0$.

and is an eigenstate of the total Hamiltonian[1]

$$\begin{aligned} H &= H_0 + H' - 2\mathscr{E}_0 \\ H' &= H_a' + H_b' = \int d^3k\, a_\alpha(\mathbf{k})[iV_{\alpha a}(\mathbf{k}) + e^{i\mathbf{k}\cdot\mathbf{r}_0} iV_{\alpha b}(\mathbf{k})] + \text{h.c.} \end{aligned} \tag{21.9}$$

in the limit $\mathbf{r}_0 \to \infty$. We shall now simply calculate the ground-state energy

$$\langle \xi_a, \xi_b, \mathbf{r}_0 |H| \xi_a, \xi_b, \mathbf{r}_0\rangle = \mathscr{V}(\mathbf{r}_0)$$

To refine this procedure, we have to use a more refined trial state which also contains admixtures of "nucleon + incoming pion" states.[1] They describe the distortion of the pion cloud and correspond to diagrams of the type shown in Fig. 21.5.

Remembering

$$(H_0 + H' - 2\mathscr{E}_0)R_a \mid \xi_a, \xi_b, \mathbf{r}_0) = (H_b' - \mathscr{E}_0)R_a \mid \xi_a, \xi_b, \mathbf{r}_0) \tag{21.10}$$

we obtain

$$\begin{aligned} &HR_aR_b \mid \xi_a, \xi_b, \mathbf{r}_0) \\ &\quad = \{[[H,R_a], R_b] + R_aH_a'R_b + R_bH_b'R_a - R_aR_b(H + 2\mathscr{E}_0)\} \mid \xi_a, \xi_b, \mathbf{r}_0) \end{aligned} \tag{21.11}$$

The double commutator is zero to the order considered. It is of shorter range ($\lessdot \frac{1}{2}$) and contributes to the next higher order (two-meson exchange) of the potential.[2] Furthermore, we have

$$\begin{aligned} &(H + 2\mathscr{E}_0) \mid \xi_a, \xi_b, \mathbf{r}_0) \\ &\quad = \int d^3k\, [-a_\alpha^\dagger(\mathbf{k})iV_{\alpha a}(\mathbf{k}) - ie^{-i\mathbf{k}\cdot\mathbf{r}_0}V_{\alpha b}(\mathbf{k})a_\alpha^\dagger(\mathbf{k})] \mid \xi_a, \xi_b, \mathbf{r}_0) \end{aligned} \tag{21.12}$$

since all other terms contain destruction operators on the right-hand side. Because $a^\dagger V_a$ commutes with R_b and $a^\dagger V_b$ with R_a, the two terms of (21.12) can be canceled against corresponding terms in (21.11), and we get[3]

$$\begin{aligned} \langle \xi_a, \xi_b, \mathbf{r}_0 |H| \xi_a, \xi_b, \mathbf{r}_0\rangle = \sum_\alpha \int d^3k[&\langle \xi_a \mid V_{\alpha a}(\mathbf{k}) \mid \xi_a\rangle \langle \xi_b, \mathbf{r}_0 \mid a_\alpha(\mathbf{k}) \mid \xi_b, \mathbf{r}_0\rangle \\ &+ \langle \xi_b, \mathbf{r}_0 \mid V_{\alpha b}(\mathbf{k}) \mid \xi_b, \mathbf{r}_0\rangle\langle \xi_a \mid a_\alpha(\mathbf{k}) \mid \xi_a\rangle e^{i\mathbf{k}\cdot\mathbf{r}_0}] \end{aligned} \tag{21.13}$$

---

[1] The exact form of $V_\alpha(\mathbf{k})$ is given by (19.1), and the subscripts a, b imply that $\boldsymbol{\sigma}_a$, $\boldsymbol{\tau}_a$ or $\boldsymbol{\sigma}_b$, $\boldsymbol{\tau}_b$ are to be taken. The abbreviation h.c. means hermitian conjugate.

[2] See R. E. Cutcosky, *Phys. Rev.*, **112**:1027 (1958) and **116**:1272 (1959); Iu. I. V. Novozhilov, *J. Exptl. Theoret. Phys.* (*U.S.S.R.*), **32**:1262 (1957) and **33**:901 (1957) [trans. in *Soviet Phys. JETP*, **5**:1030 (1958) and **6**:692 (1958)].

[3] The factorization $R_aR_b| \xi_a, \xi_b) = | \xi_a \rangle | \xi_b \rangle$ is possible since $| \xi_a, \xi_b)$ is the direct product of $| \xi_a)$ and $| \xi_b)$.

The expectation value of $a_\alpha(\mathbf{k})$ is given by (19.2), and we finally obtain

$$\mathscr{V}(\mathbf{r}_0) = -\frac{f_r^2}{(2\pi)^3}\int \frac{d^3k}{\omega^2}\, e^{i\mathbf{k}\cdot\mathbf{r}_0}\boldsymbol{\sigma}_a\cdot\mathbf{k}\,\boldsymbol{\sigma}_b\cdot\mathbf{k}\sum_\alpha \tau_{a\alpha}\tau_{b\alpha}\rho^2(k) \tag{21.14}$$

which, as predicted, is exactly the potential (21.4) or (21.7) with the coupling constant f^2 replaced by the renormalized one f_r^2.

21.3. Comparison with Experiment. Having found the exact value for the strength of the potential, we are in a position to make a quantitative comparison with experiment. To this end we have to take into account the translational degrees of freedom of the nucleons and add their kinetic energy to the potential (21.7). The two-nucleon wave function $\chi_{\xi_a\xi_b}(\mathbf{r}_0)$ is defined by writing the two-nucleon state

$$|\,2\rangle = \sum_{\xi_a\xi_b}\int d^3r_0\chi_{\xi_a\xi_b}(\mathbf{r}_0)\,|\,\xi_a, \xi_b, \mathbf{r}_0\rangle \tag{21.15}$$

If $|\,2\rangle$ is an eigenstate of the total energy, then χ obeys the Schrödinger equation

$$[-\delta_{\xi_a\xi_b,\xi_a'\xi_b'}\frac{1}{M}\nabla^2 + \mathscr{V}_{\xi_a\xi_b,\xi_a'\xi_b'}(\mathbf{r}_0)]\,\chi_{\xi_a'\xi_b'}(\mathbf{r}_0) = E\chi_{\xi_a\xi_b}(\mathbf{r}_0) \tag{21.16}$$

To solve this equation, we have to diagonalize $\mathscr{V}$ in spin and isospin space. This is carried out in most books[1] on nuclear physics and leads from the deuteron ground state to the well-known equations which couple S and D states. Since our potential (21.11) describes only the long-range part of the force, it has to be adjusted for $r_0 < 1$, in a partially phenomenological manner. However, the quadrupole moment and the binding energy of the deuteron are sensitive principally to the behavior of the wave function χ at distances $r_0 > 1$ and should be fitted by the potential (21.14) with the coupling constant found from pion-nucleon scattering. A detailed discussion[2] of (21.16) shows that unless the coupling constant is within the range $0.065 \lessdot f_r^2/4\pi \lessdot 0.09$, it is very difficult to fit the above static properties of the deuteron. The low-energy scattering parameters, that is, the scattering length and effective ranges in S and P states, are also consistent with the solution of (21.16) for $r_0 > 1$.

Recently ways have been found to extract directly from the nucleon-scattering data that part which comes from the exchange of one pion. The method consists in extrapolating the scattering cross section to unphysical energies and angles where the intermediate pion becomes real. In this unphysical situation it can propagate over large distances, and the scattering amplitude has a pole. The residue at this pole is directly related to the renormalized coupling constant, and the experimental data[1] give $f_r^2/4\pi = 0.07 \pm 0.01$, in agreement with the value deduced by other processes.

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[1] See, e.g., J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," chap. 2, John Wiley & Sons, Inc., New York, 1952.

[2] See J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, *Suppl. Progr. Theoret. Phys. (Kyoto)*, 3:32 (1956).

At higher energies the situation becomes more complicated, since velocity-dependent forces such as a spin-orbit force enter on the scene. Such forces are obviously outside the scope of the static model. Correspondingly, application of the latter to processes like pion production in nucleon-nucleon collisions is dubious, since at energies above the threshold for this process the picture cannot be described by a simple static model.

21.4. Concluding Remarks. In summary, we can say that the static model is a theory with a reasonably transparent mathematical structure. It is remarkably successful in tying together data concerning various low-energy mesonic phenomena. The fact that the renormalized coupling constants determined from pion scattering and photoproduction and from nuclear forces agree within 15 per cent shows that quantum field theory is capable of penetrating into the subnuclear world. But one should not be blinded by this success. The model is an obvious simplification of the true state of affairs. A more accurate theory must also treat nucleons as quantized fields and has to face the complications of relativity. Unfortunately, not only does this mean additional computational difficulties, but the whole mathematical structure of such theories is unknown.[2]

Disregarding these questions and guided by perturbation theory, we can postulate a simple analytic behavior for the scattering amplitude. Using the resulting so-called Mandelstam representation,[3] we can deduce equations[4] for the scattering amplitude which are generalizations of the Low equation we discussed. They contain not only all kinematical corrections but also antinucleon-nucleon pairs, pion-pion interactions, etc. There ensue further branch lines and new channels at higher energies. Since the coupling of the three internal pion states to the nucleon already makes an exact solution of the Low equation impossible in the static model, it is clear that these equations are exceedingly complicated. However, neglecting meson-meson interactions and going to the limit of $M \to \infty$, we obtain the equations of the static model, when this limit exists. In this way the large effects which are correctly predicted by the static theory follow from a more fundamental approach. It is even hoped that in this way medium-sized effects can be unambiguously calculated. Certainly much further work is required before all the complications of relativity and quantum theory, taken together, are worked out.

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[1] See M. J. Moravcsik in G. R. Screaton (ed.), "Dispersion Relations," p. 117, Scottish Universities' Summer School, 1960, Oliver and Boyd, Edinburgh, 1961; and P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, *Phys. Rev.*, **114**:880 (1959).

[2] Except for one unrealistic case. See W. Thirring, *Ann. phys.*, **9**:91 (1958), and *Nuovo cimento*, **9**:1007 (1958); V. Glaser, *Nuovo cimento*, **9**:1005 (1958).

[3] See S. Mandelstam, *Phys. Rev.*, **115**:1741 (1959) and **115**:1752 (1959).

[4] See G. F. Chew in G. R. Screaton (ed.), "Dispersion Relations," p. 167, Scottish Universities' Summer School, 1960, Oliver and Boyd, Edinburgh, 1961.

Further Reading (on potentials)

M. Taketani, S. Machida, and S. Onuma, *Progr. Theoret. Phys.* (*Kyoto*), **7**:45 (1952).
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E. M. Henley and M. A. Ruderman, *Phys. Rev.*, **92**:1036 (1953).
S. Gartenhaus, *Phys. Rev.*, **100**:900 (1955).
P. S. Signell and R. E. Marshak, *Phys. Rev.*, **106**:832 (1957) and **109**:1229 (1958).

Appendix

The formulas compiled below express various quantities in the different representations used in the text. They are all equally important and exhibit different features of the theory. Coordinate space has the most intuitive appeal and was first used with cubic or spherical boundary conditions. These are not realized in actual experiments and are considered only a mathematical aid. Correspondingly, we quickly passed to an infinite volume and specified certain initial conditions. The expansion in plane waves diagonalizes energy, momentum, and charge. In problems with a spherical source, the momentum of the field is no longer a constant, but the angular momentum is. In this case a spherical-wave expansion reduces the problem further. Several conventions are used for defining the field variables in these various representations. We choose those for which the commutator of the field operators is just equal to a Kronecker δ for discrete variables and a Dirac δ function for continuous ones. The advantage of this convention is that in the expressions for energy, momentum, etc., there is a sum or integral without further numerical factors. This is achieved at the expense of having several factors in the expression of the local field in terms of these variables.

r space	Plane waves	Spherical waves
	Discrete: $\mathbf{k} = (n_x, n_y, n_z)\dfrac{2\pi}{L}$ Continuum: $\dfrac{1}{L^3}\sum_{\mathbf{k}} \to \int \dfrac{d^3k}{(2\pi)^3}$	Discrete: $k = \left(n + \dfrac{l}{2}\right)\dfrac{\pi}{L}$ Continuum: $\dfrac{1}{L}\sum_{k} \to \dfrac{1}{\pi}\int_0^\infty dk$
Field		
$\phi(\mathbf{r})$	Discrete: $\sum_{\mathbf{k}} \dfrac{a_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}{(2\omega L^3)^{\frac{1}{2}}} + \text{h.c.} \qquad (4.8)$ Continuum: $\int \dfrac{d^3k\, a(\mathbf{k})}{(2\pi)^{\frac{3}{2}}(2\omega)^{\frac{1}{2}}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{h.c.}$	Discrete: $\sum_{k,l,m} Y_l^m(\theta,\varphi) U_k^l(r) e^{-i\omega t} \dfrac{a_{kl}^m}{(2\omega L/\pi)^{\frac{1}{2}}} + \text{h.c.}$ Continuum: $\int_0^\infty dk \sum_{l,m} Y_l^m(\theta,\varphi) U_k^l(r) e^{-i\omega t} \dfrac{a_l^m(k)}{(2\omega)^{\frac{1}{2}}} + \text{h.c.}$
Commutation relations		
$[\phi(\mathbf{r},t), \dot{\phi}(\mathbf{r}',t)] = i\delta^3(\mathbf{r}-\mathbf{r}')$	Discrete: $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}$ Continuum: $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}')$	Discrete: $[a_{kl}^m, a_{k'l'}^{\dagger m'}] = \delta_{m,m'}\,\delta_{l,l'}\,\delta_{k,k'}$ Continuum: $[a_l^m(k), a_{l'}^{\dagger m'}(k')] = \delta_{m,m'}\,\delta_{l,l'}\,\delta(k-k')$

Energy		
$\int d^3r\,\frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2]$	$\sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\omega = \int d^3k a^{\dagger}(\mathbf{k}) a(\mathbf{k})\omega$	$\sum_{l,m,k} a_{kl}^{\dagger m} a_{kl}^{m}\omega = \int dk \sum_{l,m} a_l^{\dagger m}(k) a_l^m(k)\omega$
Momentum		
$-\int d^3r\,\dot{\phi}\nabla\phi$	$\int d^3k\,a^{\dagger}(\mathbf{k})a(\mathbf{k})\mathbf{k}$	Complicated
Angular momentum		
$\mathbf{L} = -\frac{1}{2}\int d^3r\,\dot{\phi}(\mathbf{r} \times \nabla\phi) - \frac{1}{2}\int d^3r\,(\mathbf{r} \times \nabla\phi)\dot{\phi}$	$-i\int d^3k\,a^{\dagger}(\mathbf{k})\mathbf{k} \times \nabla_{\mathbf{k}} a(\mathbf{k})$	$L_z = \int_0^{\infty} dk \sum_{l,m} a_l^{\dagger m}(k) a_l^m(k) m$
Interaction Hamiltonian for π mesons in static model		
$\frac{f}{\mu}\int d^3r\,\rho(\mathbf{r})\tau_\alpha \boldsymbol{\sigma} \cdot \nabla\phi_\alpha$	$i\frac{f}{\mu}\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\rho(\mathbf{k})\tau_\alpha \boldsymbol{\sigma} \cdot \mathbf{k}\frac{a_\alpha(\mathbf{k})}{(2\omega)^{\frac{1}{2}}} + \text{h.c.}$	$\frac{f}{\mu}\sum_{m,\alpha}\int_0^{\infty}\frac{dk\,k^2}{(12\pi^2\omega)^{\frac{1}{2}}}\rho(\mathbf{k})\tau^{\alpha}\sigma^m a_1^{\alpha m}(k) + \text{h.c.}$

List of Symbols

There are a few places in the text where the same symbol is used to denote two different physical quantities. We hope that in these cases the choice will be clear to the reader from the context in which the symbol is used.

The numbers in parentheses denote the section in which the symbol first appears.

General Notation

$\dagger$, h.c.	Hermitian conjugate	(2.1)
$*$, c.c.	Complex conjugate	(1.2)
$\dot{}$, $\partial/\partial t$	Time derivative	(1.2)
$\vert\,)$	Bare states	(8.1)
$\vert\,\rangle$	Physical states	(8.1)
i, j, etc.	Latin subscripts (and superscripts) denote space components	(15.1)
α, β, etc.	Greek subscripts (and superscripts) denote charge-space (isospin) components	(15.1)
$\underset{\mathbf{k}}{\mathsf{S}}$	$L^{-3} \sum_{\mathbf{k}} = (2\pi)^{-3} \int d^3k$	(4.1)

Greek Symbols

α	Destruction operator for positively charged field quanta	(7.1)
β	Destruction operator for negatively charged field quanta	(7.1)
Γ	Width of resonance	(12.2)
$\Gamma^{(+)}(\mathbf{r})$	$\underset{\mathbf{k}}{\mathsf{S}}\, \omega e^{i\mathbf{k}\cdot\mathbf{r}}$	(5.4)

Greek Symbols

γ	Photon	(20.2)
Δ	Small region in momentum space	(10.1)
Δ^{adv}	Advanced Green's function	(8.1)
Δ^{ret}	Retarded Green's function	(8.1)
$\delta(\omega)$, $\delta(k)$	Phase shift	(8.2)
ϵ	Infinitesimally small positive quantity	(11.1)
$\boldsymbol{\epsilon}$	Polarization vector of electromagnetic field	(20.2)
ϵ_{ijk}	Totally antisymmetric tensor	(5.1)
η	Probability	(6.3)
η_{fi}	Transition probability	(8.2)
θ	Angle in expansion of field	(5.1)
	Constant in classical meson theory	(16.1)
ϑ	Scattering angle	(16.2)
Λ	Transformation matrix in charge space	(7.2)
λ	Wavelength	(5.4)
	Dimensional coupling constant	(11.1)
	Undetermined multiplier	(17.4)
λ_r	Renormalized dimensional coupling constant	(12.2)
$\lambda^{(u)}$	Eigenvalues of the Born-approximation scattering matrix	(18.5)
μ	Mass of pion	(15.1)
$\boldsymbol{\mu}_N$	Part of nucleon magnetic moment	(20.1)
ν	Frequency	(5.4)
	Number of particles in a finite volume	(6.3)
ξ	Combined spin-isospin indices for nucleon	(17.1)
π	Momentum-density operator	(4.2)
	Pion	(7.2)
$\rho(\mathbf{r})$	Source distribution in space	(8.1)
$\rho_{\mathbf{k}}$, $\rho(\mathbf{k})$	Source distribution in momentum space	(9.2)
σ	Cross section	(8.2)
$\boldsymbol{\sigma}$	Spin operator	(15.1)
τ	2 × 2 isospin matrices	(13.1)
$\bar{\tau}$	Isospin operator when it operates on the proton in the Lee model	(13.3)
ϕ	Klein-Gordon field (operator)	(4.1)
$\phi^{(+)}$	Positive-frequency part of ϕ ($\propto e^{-i\omega t}$)	(5.3)
$\phi^{(-)}$	Negative-frequency part of ϕ ($\propto e^{i\omega t}$)	(5.3)
$\phi_{\pm}$	Charged field operators	(7.1)
ϕ^{in}	Field at $t = -\infty$	(8.1)
ϕ^{out}	Field at $t = +\infty$	(8.1)
φ	Azimuthal angle	(5.1)
χ	$g\rho_{\mathbf{k}}(2\omega^3L^3)^{-\frac{1}{2}}$	(9.3)
	Two-nucleon relative wave function	(21.3)
ψ	Schrödinger field operator	(4.2)

Greek Symbols

$\bar{\psi}$	Schrödinger field operator when it acts on the proton in the Lee model	(13.3)
ψ_E	Eigenfunction of energy E	(2.1)
ψ_0	Ground-state eigenfunction	(2.1)
Ω	Harmonic-oscillator frequency	(1.2)
	Wave matrix	(11.1)
ω	Circular frequency and energy	(2.1)
$\bar{\omega}$	Matrix for ω	(11.1)
ω_r	Resonant energy	(12.2)
ω_s	Bound-state frequency	(16.1)

Other Symbols

A	Destruction operator for ϕ^{in}	(8.1)
	Amplitude of plane wave	(16.2)
	Matrix	(17.5)
$\mathbf{A}$	Electromagnetic vector potential	(7.1)
a	Distance between atoms	(1.2)
	Field-quantum destruction operator	(2.1)
	Radius of source	(16.1)
$\mathscr{A}$	Scattering amplitude	(14.1)
B	Destruction operator for ϕ^{out}	(8.1)
	Matrix	(17.5)
$C(\omega)$	Constant in classical meson theory	(16.1)
C_{lt}	Normalization constant	(17.1)
D	$\pm$ Im $D_\pm$	(11.2)
$D_\pm$	Denominator in wave matrix	(11.1)
D_1	Re $D_\pm$	(11.2)
$D^{(r)}$	Renormalized D	(12.2)
d	Displacement of harmonic oscillator	(2.3)
E	Energy	(2.2)
E_0	Ground-state energy	(2.2)
E_k	$k^2/2m$, energy of Schrödinger field quantum	(4.2)
E_i	Energy of state $\|\text{out}, i\rangle$	(14.2)
$\mathscr{E}_0$	Energy shift due to interaction (ground-state energy of system)	(9.2)
$\mathscr{E}_0(N)$	Ground-state energy of N sources	(9.5)
$\mathfrak{E}$	Electric field intensity	(10.2)
ΔE	Excitation energy of first excited state	(17.4)
$F(\mathbf{r},t)$	Wave function of field quantum	(6.1)
$F_{\mathbf{k}}$, $F(\mathbf{k})$	Momentum-space wave function of field quantum	(6.1)
F	Weighting function of t matrix	(18.3)
F_v, F_s	Form factors	(20.1)
$f(\mathbf{r},t)$	Wave function of field quantum	(6.1)

Other Symbols

$f_{\mathbf{k}}, f(\mathbf{k})$	Momentum-space wave function of field quantum	(6.1)
f	Meson-nucleon dimensionless coupling constant in symmetrical pseudoscalar theory	(15.1)
$\bar{f}$	Reduced meson-nucleon coupling constant	(17.4)
f_r	Renormalized meson-nucleon coupling constant	(18.4)
$\mathfrak{F}$	Weighting function of g_u	(18.7)
$G(\mathbf{r},t)$	Green's function	(8.1)
G	Weighting function of t matrix	(18.3)
$g(\mathbf{k},K_0)$	Four-dimensional Fourier transform of the Green's function	(8.1)
g	Coupling constant	(9.1)
g_r	Renormalized coupling constant	(13.6)
g_u	Function proportional to $1/h^{(u)}$	(18.7)
$\mathfrak{G}$	Weighting function of g_u	(18.7)
H	Hamiltonian	(1.3)
$H^{(1)}$	Hankel function of the first kind	(5.4)
$H(\mathbf{r})$	Local-energy-density operator	(6.1)
$h^{(u)}$	Projected amplitude of t matrix	(18.5)
$\mathscr{H}$	$H - \mathscr{E}_0$	(9.2)
$\mathfrak{H}$	$-\ \lvert\rho(k)\rvert^2/[(2\pi)^3T(k)]$ and analytic continuation of this function	(14.2)
I	Intensity	(6.3)
$\mathbf{J}$	Total angular momentum	(16.1)
j_l	Spherical Bessel function	(5.1)
$\mathbf{j}$	Current operator	(7.1)
K	Abbreviation for j, α, k indices	(18.2)
K_0	Conjugate variable to time	(8.1)
k	Wave number, momentum	(1.3)
k_r	Resonant momentum	(12.2)
$k_{\max}$	Source cutoff momentum	(15.3)
L	Length of line	(1.3)
	Lagrangian	(4.2)
$\mathbf{L}$	Orbital angular momentum operator	(5.1)
$L_n(\lambda)$	A certain function	(17.5)
l	Angular momentum	(5.1)
$\mathscr{L}$	Lagrangian density	(4.2)
M	Physical mass of source	(9.2)
M_0	Mechanical mass of source	(9.2)
M_1, M_2, M	Matrices	(12.4)
ΔM	Energy of physical neutron in the Lee model	(13.3)

Other Symbols

m	Mass of field quantum	(4.1)
	z component of angular momentum	(5.1)
$\mathscr{M}$	Magnetic moment of nucleon	(19.5)
$\mathfrak{M}$	Reduced magnetic moment of nucleon	(19.5)
N	Number of atoms	(1.2)
	Number of modes	(3.1)
	Number of particles operator	(5.3)
	Nucleon	(14.1)
$N_{\pm}$	Number of $\pm$ charged particles operator	(7.1)
$N(\mathbf{r})$	Particle-density operator	(5.4)
n	Quantum number of state	(2.1)
	Occupation number	(3.1)
	Number of field quanta	(5.3)
	Neutron	(13.1)
$\mathbf{n}$	Unit normal	(2.3)
$\mathscr{N}$	Normalization constant	(17.3)
$\mathscr{O}$	Arbitrary operator	(2.3)
$\mathbf{P}$	Momentum operator	(5.1)
$\mathbf{P}(\mathbf{r})$	Momentum-density operator	(6.1)
P_s	Normal momentum coordinate	(1.2)
p	Momentum	(1.2)
	Proton	(13.1)
$\mathscr{P}$	Parity operator	(5.2)
$\mathfrak{P}$	Projection operator	(8.2)
Q	Charge operator	(7.1)
	Diagonal matrix of $q_{\alpha j}$ in strong-coupling meson theory	(17.5)
$Q(\mathbf{r})$	Charge-density operator	(7.1)
Q_s	Normal coordinate	(1.2)
q	Displacement	(1.2)
$\bar{q}$	Mean value of q	(2.3)
Δq	Zero-point fluctuation of q	(2.3)
$\mathscr{Q}$	Nonhermitian operator	(4.2)
R	Radius of boundary	(5.1)
	Scattering matrix	(11.3)
	Ground-state contribution to T matrix	(18.3)
$\mathbf{R}$	Center of mass or of energy coordinate	(5.3)
$R(a^{\dagger})$	General operator that relates a physical nucleon to a bare one	(17.1)
$\mathbf{r}$	Vector spatial coordinate	(2.3)
r_u	Effective range	(18.6)
r_0	Separation between two nucleons	(21.1)
$\mathfrak{r}_1$, $\mathfrak{r}_2$	Renormalization constants	(17.1)
S	S matrix	(8.2)

Other Symbols

$\mathbf{s}$	Spin of nucleon	(15.1)
T	Scattering matrix	(8.2)
	Total isospin	(15.2)
$\mathcal{T}$	Rotation matrix in charge space	(7.2)
t	Time	(1.2)
	Generalized isospin operator in n-dimensional charge space	(7.2)
$\mathbf{t}$	Meson isospin operator	(15.2)
$t(k)$, $t(z)$	Reduced scattering matrix	(14.3)
U	Unitary operator	(7.1)
$U_k^l(r)$	Radial function in the expansion of $e^{i\mathbf{k}\cdot\mathbf{r}}$	(5.1)
U_b	Bound-state wave function	(11.3)
V	Electrostatic potential	(7.1)
$V(\mathbf{r},t)$	Generalized potential-type source density	(8.1)
V_K	Radial part of pion source term	(18.2)
$V_\alpha(\mathbf{k})$	Pion source term	(19.1)
v	Velocity	(1.3)
$\mathcal{V}$	Static potential between two nucleons	(21.1)
W_{fi}	Transition rate from state i to f	(8.2)
W_i	Total transition rate out of the state i	(8.2)
W	Reduced energy	(17.4)
w	$m + k^2/2m$, energy of quantum in the Lee model	(13.2)
Y_l^m	Spherical harmonic	(5.1)
$Y(r)$	Extended Yukawa-type potential	(19.1)
Z	Coupling-constant renormalization factor	(13.4)

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